MONTE CARLO SIMULATION OF UTILITY FUNCTION SHAPES

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Abstract

Utility is one of the basic components of decision-making. While used to make decisions under risk, it is also an approach of multi-criteria decision-making. The principles of the utility theory are used for value functions. The utility function can adopt three basic shapes: linear, concave and convex. Each shape describes a different approach of the decision maker towards the risk. For maximizing criteria as profit, the concave shape represents a risk averse attitude of the decision maker, whereas, the convex represents risk-prone decision maker. At the end of the decision-making process, the decision is made based on the utility. The paper compares the three basic shapes of utility and their influence on decisions using the Monte Carlo simulation. Different ways to compute the utility are discussed and compared. Computation of the utility is an alternative approach to the commonly used subjective elicitation of the utilities. Advantages, disadvantages and usefulness of the approaches are discussed.

Key words: Utility function, Monte Carlo simulation, shapes of utility function, utility computation

JEL Code: D81, G32

Introduction

Definitions of utility usually contain words as satisfaction, usefulness, consumption of goods and services, and meeting the needs and wants. Utilities are used for various purposes in various fields. A typical application of the utilities is in microeconomics to derive demand.

Theories differentiate cardinal and ordinal utilities. Cardinal utilities are quantified, usually in the range from 0 to 1 while the ordinal utilities allow individuals to only order items according to the preference without quantification. Cardinal utility theory is assumed in the paper.

Utilities express the preferences of decision makers and show their directions. One of the basic concepts based on utilities is utility maximization. Individuals do things to maximize their utilities.

Explanations of individuals´ behaviour via utility have been criticised over its history. The criticism concentrates on the quantification of utilities as well as on the utility maximization. Sedlacek starts with definition from Collins dictionary of Economics: „*Utility: satisfaction or pleasure that an individual derives from the consumption of good or service*.“ He argues that individuals do not maximize utilities all the time – they sleep, have fun etc. They do not consume goods and services nonstop. Adding all individuals´ activities to the utility definition does not satisfy Sedlacek. The definition would be “*Individuals do what they want to do*”. Sedlacek continues with arguments against utility maximization with the example of Jan Hus who was burnt to death for his believes. Did Jan Hus maximize his afterlife utility? (Sedláček, 2009)

Despite the criticism, the utility is very useful tool to describe individuals´ behaviour and to deal with decisions.

1 Utility Function

Utility functions represent the utility of individuals in the cardinal utility theory.

1.1 Utility in decision-making

In the field of decision-making, the utility function represents a model of an individual´s attitude towards risk; it embodies an important fundamental trade-off: monetary return versus riskiness.

The decisions under risk are based on the expected utility. The expected utilities are usually combined with methods such decision matrixes, decision trees and probability trees. These methods incorporate risk factors into decisions. Other ways to make decisions exist. Compared to the expected utility, the expected value does not contain the attitude towards risk.

The next great advantage of utilities in decision-making is the possibility to combine more aspects into one utility value. Criteria in multicriteria decision-making are usually non-additive. The solution is in their transformation into one criterion. Many methods of multicriteria decision-making overcome this difficulty using utilities. Keeney and Raiffa present a multiattribute utility theory (MAUT). They distinguish between additive utility function and multiplicative utility function (Keeney & Raiffa, 1976).

Value functions are very similar in its use to deal with multiattribute decision problems. Value functions are used for decisions under certainty whereas utility functions for decisions under risk and uncertainty (Goodwin & Wright, 2009). Assessing value functions differs from assessing utility functions. Interpretation of value functions is also different from the interpretation of utility functions. Nevertheless, the principles of making decisions based on the utility and value functions are very similar. In the paper, the term utility function covers also the term value function. Both, utilities and values indicate the desirability.

In the paper, the additive utility theory is used to calculate the total utility. The partial utility functions are summed up to the total utility of an alternative.

1.2 Shapes of utility functions

The theory distinguishes three basic shapes of utility functions: concave, convex and linear. Figure 1 presents the three shapes for the case of maximizing criterion. With maximizing criterion, individuals prefer higher values over lower values, profit and number of customers are examples of maximizing criterion. In contrast, lower values are preferred over higher with minimizing criterion as depicted in Figure 2. Minimizing criteria are costs or number of complaints.

Fig. 1: Shapes of utility for maximizing criterion

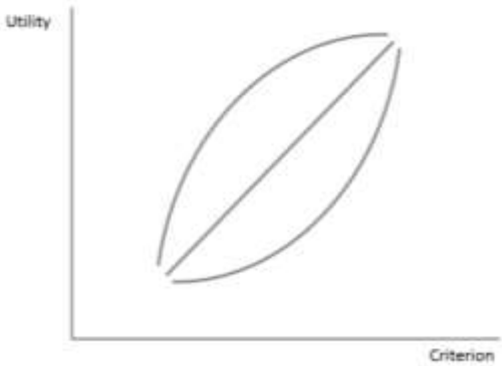
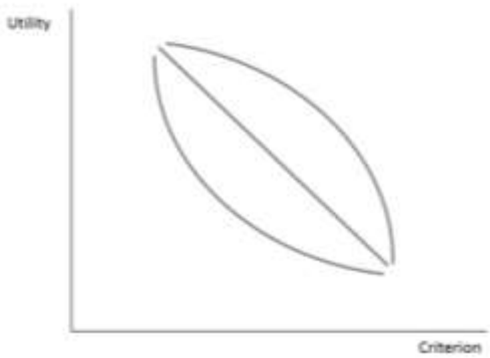


Fig. 2: Shapes of utility for minimizing criterion



The different shapes represent different attitudes towards risk: concave – risk-averse; convex – risk-seeking; linear – risk-neutral.

Different attitudes towards risk mean that individuals are willing to accept different levels of risk. Risk-neutral decision makers do not care about the risks. The decisions based on the expected utility is then the same as decision based on the expected value.

Figures 1 and 2 present the basic shapes. The real utility function can deviate. The level of risk-aversion changes the concavity of the function. The shape can also change from convex to concave with the change in the criterion values. Decision makers search the indifference points (Stewart, 1993). Research proves the differences in utility shapes among managers and shows its link to organizational behaviour (Pennings & Smidts, 2003). While making decisions, individuals have to assign utilities to positive values as well as to negative values.

Basic shapes of value function are the same as the utility functions pictured on Figures 1 and 2. The interpretation differs. The value functions do not incorporate risk. The linear shape of value functions represent situations where increases in the value of the criterion add the same value to the individuals. The concave value functions describe the criterion where the increases in the criterion are progressively smaller. Concave functions are suitable for criteria as office areas. Convex value functions are suitable for criteria where the absolute value comes with completed set like collected items.

Operation with linear utility functions is much easier than with non-linear functions (Andre & Riesgo, 2007). Linear models are acceptable in many cases (von Winterfeldt & Edwards, 1986). The goal of this paper is to investigate the differences between the shapes.

1.3 Utility assessment

Assessing utility functions is a matter of subjective judgment. The theory offers various approaches to elicit the utility / value functions. Some authors recommend discussion between the decision analyst and the decision maker, some offer approaches based on the certainty equivalents or on probabilities. Andre proposes a method to elicit non-linear multiattributed utility functions based on the data envelopment analysis (DEA) (Andre, 2009).

The correctness of utility functions cannot be mathematically tested. The functions are subjective. More questioning can be used to validate utility functions. (Keeney R., 1973) Using the utility functions, decision makers should fulfil the axioms related to the utilities. The axioms such as ordering and transitivity, continuity etc. create the assumptions for correct use (Clemen & Reilly, 2001).

1.4 Computation of utility

Computation of utilities is an alternative approach to subjective utility elicitation. Different methods exist to compute utilities of the criterion values. Decision makers choose the formula closest to their preferences. The formulas result in convex, concave or linear utilities as well as elicited utilities.

The values of the criterion are transformed at first to range from 0 to 1. Formula chosen for the transformation is:

$$x_i = \frac{x'_i - x'_0}{x'^* - x'_0}, \quad (1)$$

where the x'_i denotes the original value, x'^* the best original value and x'_0 the worst original value of the criterion.

The formula used to transform the original values into normalized values is not the only formula available. Advantage of the formula is the usability for maximizing criteria as well as for minimizing criteria and for positive values as well as negative values.

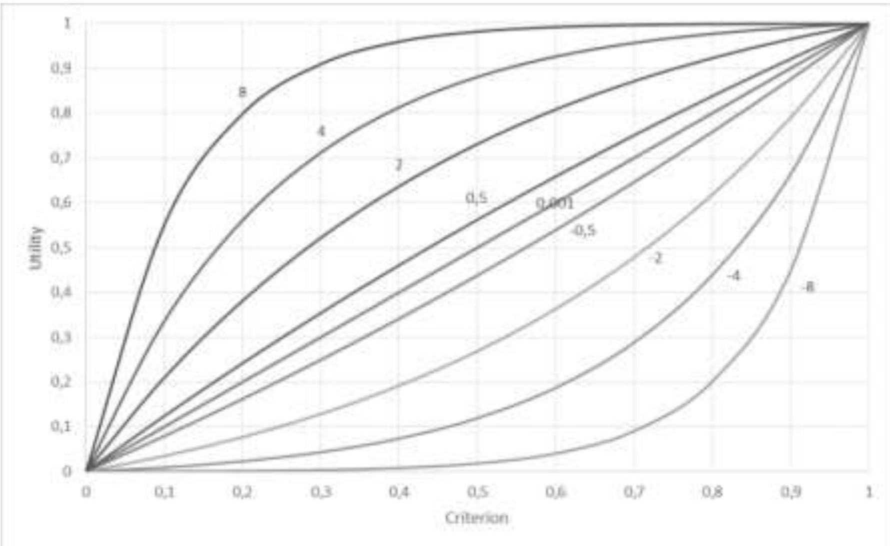
In the next step, utilities are calculated using formula:

$$u_i(x_i) = \frac{1 - \exp(-cx_i)}{1 - \exp(-c)}. \quad (2)$$

The constant c measures the curvature of the utility function. Positive c constants represent concave shapes, negative represent convex utility function. The constants c close to 0 yield in linear function. (Lahdelma & Salminen, 2012).

In the paper, ten different constants c are used: -8; -4; -2; -0.5; -0.001; 0.001; 0.5; 2; 4; 8. Figure 3 shows the shapes of utility functions for each c value.

Fig. 3: Shapes of computed utility functions



2 Monte Carlo Simulation

Monte Carlo simulation is used to investigate the results of different utility function shapes. Simulation is performed in the software @RISK. Monte Carlo simulation is computerized mathematical technique that generates values of input variables and calculate values for output values.

The model consists of input and output variables. The output variables are the total utilities of the alternative based on the expected utilities. Additive utility function is used. The additive utility function is easier to construct and understand compared to the multiplicative function (Belton & Stewart, 2003). The total utilities of alternatives are based on the formula:

$$U(x) = \sum w_i u_i(x_i) p_i. \qquad (3)$$

The total expected utilities of an alternative are sums of products of partial utilities of criteria and the weighting factors with respect to the probabilities of the scenarios. In the paper, the same importance is assigned to the criteria and the same probabilities to all scenarios.

The model consists of four criteria. Consequences of criterion 1 are generated with the Monte Carlo simulation. The Beta Distribution is chosen for the criterion with the minimum 70, maximum 130, mean 100 and standard deviation 13.416. The distribution is symmetrical. 90% of the values lie between 78.1 and 121.9.

Consequences for criteria 2, 3 and 4 are given in Table 1. The row Total shows the total values for criteria 2, 3 and 4 over all ten scenarios. The total value is the expected value where the probabilities of the scenarios are the same.

Tab. 1: Consequences for criteria 2, 3 and 4 for 10 scenarios

	c_2	c_3	c_4
s_1	0.868	0.056	0.347
s_2	0.922	0.052	0.841
s_3	0.641	0.590	0.410
s_4	0.860	0.795	0.353
s_5	0.633	0.284	0.183
s_6	0.666	0.686	0.547
s_7	0.423	0.651	0.442
s_8	0.242	0.474	0.891
s_9	0.735	0.047	0.234
s_{10}	0.330	0.392	0.447
Total	**0.632**	**0.403**	**0.469**

The model for Monte Carlo simulation consists of:

- input values – the values for criterion 1 for 10 scenarios with Beta distribution;
- output values – the total expected utility over the four criteria for 10 values of constant c representing different attitudes towards risk.

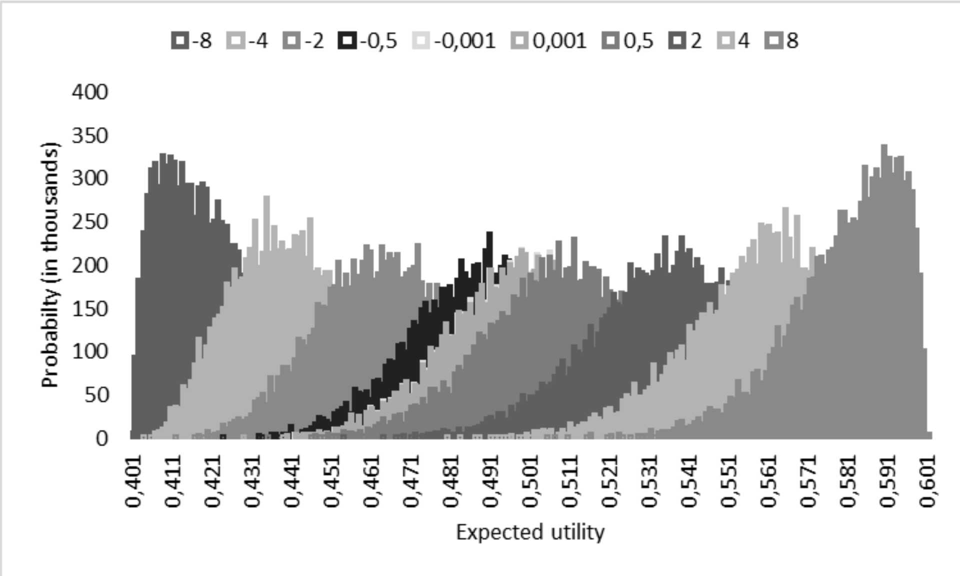
1.5 Results

The results come from a simulation with 10 000 iterations.

Figure 4 shows the probability distributions for 10 values of the c constant. The distributions are ordered as expected from the basic utility function shapes. The most risk-seeking distribution lies on the left end in the low values of expected utility. The most risk-averse reach the higher values of expected utility. The distribution for c -0.001 and 0.001 are almost similar and are in the middle of the graph.

Minimums, maximums and means are presented in Table 2. The maximum for the most risk-seeking distribution (c=-8) is 0.504. In contrast, minimum for the most risk-averse distribution (c=8) is 0.505. Where the first one ends, the second one starts. The results differ based on the utility function shapes.

Fig. 4: Probability distribution of expected utility for 10 utility function shapes



The differences in the absolute position of the probability distribution are predictable. The differences in the statistical characteristics measuring risk of the distribution shows Table 2. The distribution differs in their skewness. The input values are symmetrical for all shapes. Symmetrical input yield in non-symmetrical output. Non-symmetrical input was tested as well. The results for non-symmetrical inputs are the same, the chart of the distribution of expected utility is almost the same as on the Figure 4.

Table 2 shows also differences in the kurtosis which is also visible from the Figure 4. The skewness, kurtosis and other characteristic respond to the shape of the utility function. The more risk-averse or more risk-seeking the decision maker is, the higher skewness and the higher kurtosis. The standard deviation and variance is lower with lower c constant.

Tab. 2: Statistics of expected utility for 10 utility function shapes

	-8	-4	-2	-0.5	-0.001	0.001	0.5	2	4	8
Minimum	0.401	0.404	0.412	0.424	0.429	0.429	0.435	0.454	0.480	0.505
Maximum	0.504	0.530	0.548	0.567	0.573	0.573	0.578	0.590	0.597	0.601
Mean	0.422	0.444	0.467	0.492	0.501	0.501	0.510	0.535	0.558	0.580
Std Deviation	0.014	0.017	0.019	0.020	0.020	0.020	0.020	0.019	0.018	0.014
Variance	0.00020	0.00031	0.00037	0.00040	0.00040	0.00040	0.00040	0.00037	0.00031	0.00020
Skewness	1.037	0.632	0.354	0.114	0.031	0.031	-0.051	-0.288	-0.558	-0.954
Kurtosis	4.246	3.480	3.119	2.950	2.924	2.924	2.914	2.969	3.182	3.825

For comparison, simulation was repeated with different formula to calculate utility. The formula described previously is based on the exponential function. Authors compared the formula with formula based on power function. The formula operates with normalized values as the exponential version of the formula. The utility is the power of the normalized value.

Figure 5 shows the utility functions for both exponential and power formula. Each formula is displayed twice – for constant c equal 2 and 4 and for second and fourth power. Two “2” and two “4” functions are very close to each other.

Figure 6 depict simulation results – expected utilities over the four criteria where the first one is generated in Beta distribution according to the functions in Figure 5. Exponential with constant 2 and second power functions result in very similar probability distributions. The “4” functions differ a little – the exponential function lies in higher values of expected utility compared to power function. The difference is not significant.

Fig. 5: Comparison of formulas to calculate utility

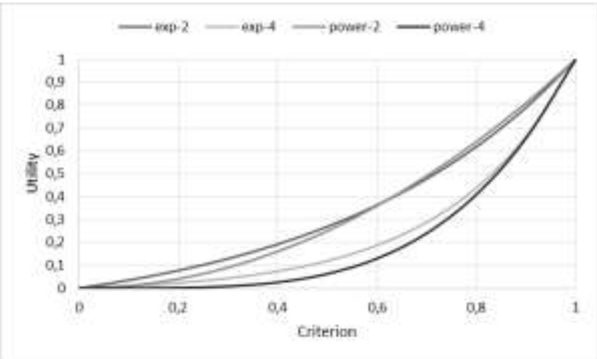
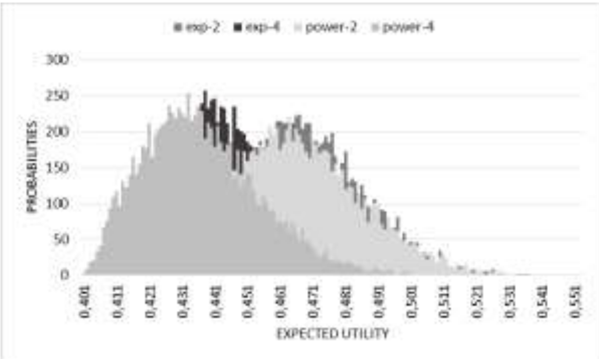


Fig. 6: Probability distributions of expected utilities for different formulas



Conclusion

Two different formulas to calculate utility / value function are presented in the paper. The results of Monte Carlo simulation for exponential formula for different attitudes towards risk show differences in probability distributions. The more extreme attitude towards risk, the more are the result values non-symmetrical, skewed and deviated.

In comparison between exponential or power based formulas, the results show very small differences. Using the power is much easier than using exponential function. Results of the simulation confirm their interchangeability.

Utilities gained with the formulas are symmetrical which might be a problem in some cases. Assigning the utilities subjectively respect the situation more than using the formulas. But the formulas are not so demanding on time and abilities of decision makers. The formulas are very good alternative to subjective elicitation and can be used for most situations.

Formulas operate with normalized values of criterion. This approach is functional for positive and negative criteria values, for maximizing and minimizing types of criteria. The utility / value function is useful in many areas of decision-making.

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