# ILL-POSED PROBLEMS IN TIME SERIES ANALYSIS

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**Abstract**

The term ill-posed problem has become popular in modern science since the middle of the 20th century. It turns out that a lot of problems not only in different areas of classical mathematics but also in astronomy, geophysics, medicine and other applied sciences can be classified as ill-posed and they belong to the most complicated. The analysis of stochastic processes provides examples of such problems. The typical example is the random walk. In fact any model generating a time series is ill posed whenever it contains a unit root.

The efficient methods for solving ill-posed problems has been developed since the end of 1950´s. They are generally denoted as regularization methods. Among them probably well-known and most favorite is the Tikhonov regularization method.

**Key words:** ill-posed problem, regularization, least squares, stationary process, random walk

**JEL Code:** C22, C65

## Introduction

Ill-posed problem is the term introduced for the first time by J. Hadamard (1902). He formulated what means that a problem is well-posed in the context of the differential equations (Cauchy problem for Laplace equation): the problem is well-posed if it has a unique solution that continuously depends on its data. Otherwise it is called ill-posed problem. Later, in 1950´s and early 1960´s, a group of Russian mathematicians led by A. N. Tikhonov appeared a lot of new approaches and some methods that became fundamental for the theory of ill posed problems and drew attention of mathematicians all over the world to this theory. Due to the powerful computers the area of applications for this theory has extended to many fields of science.

Ill-posed problems occur everywhere around us. It is clearly very easy to make a mistake if we wish to reconstruct some event in the past from several facts in the presence (causes of disease from medical examination) or vice versa to predict something in the future (one week weather forecast from atmospheric data). In fact the theory of ill-posed problems has become widely used in solving problems in a plenty of fields of science, economics not excluded (Horowitz, 2014, Hoderlein, Holzmann, 2011, Lu, Mathe, 2014). In mathematics we can find many examples of such problems almost in all branches: arithmetic, algebra, calculus, differential and integral equations, functional analysis. The ill-posed problems can be found also in the time series analysis (Sanchez, 2002). The stochastic difference equation (16) below, provides such an example. The ill-posedness is in a close relation to the problem of overdifferencing of a time series (Bell, 1987).

# 1 Ill-posed equation and its regularization

Abstract access to many problems in science and engineering leads to mathematical model the form of which is an operator equation

$$Ax = b, \tag{1}$$

where $A: U \to V$ is a mapping defined on certain sets $U$ and $V$ endowed with suitable structures. It is usual to assume that $U$ and $V$ are normed linear spaces, especially that they are Hilbert spaces, $A$ is a linear and bounded operator. The basic question is whether a solution to (1) exists and is unique in $U$ for given right side $b \in V$. Another important question is whether the solution depends continuously on the data $b$, which means that a small perturbation of $b$ is the reason for a small deviation in the solution (stability). If all these questions are answered affirmatively the problem (1) is called *well-posed* (in classical Hadamard´s sense), otherwise it is *ill-posed*. The existence and uniqueness of the solution to (1) is ensured if $A$ is a bijection (linear isomorphism). Moreover, if $U$ and $V$ are Banach spaces and $A$ is a bounded, injective operator with a closed range $\mathcal{R}(A)$ , the inverse $A^{-1}$ is bounded as well and hence the problem (1) is well-posed. The typical example of the well-posed problem is the integral equation of the second kind (Nair, 2009) or the equation for ARMA process (Arlt, 1999).

The frequent case why (1) is ill-posed is that $A$ is not bounded below. A typical example for this is a compact operator of infinite rank (Lukeš, 2012). Another example of the ill-posed problem is the topics of this paper.

## 1.1 General concept of regularization

Suppose again the equation (1), where $U$ and $V$ are normed linear spaces, $A$ is a bounded linear operator and (1) is the ill-posed problem. Now we deal with the case which often arises in practical situations, i.e. the inverse of $A$ is unbounded. The stability of the solution is obtained by approximation of the given ill-posed problem by certain well-posed one. These procedures are called regularization methods.

The *regularization strategy* is a family of bounded linear operators $R_\alpha : V \to U, \alpha > 0$, for which $\lim_{\alpha \to 0+} \|R_\alpha Ax - x\|$ for any $x \in U$. The regularization strategy may not be uniformly bounded. In fact, there is a sequence of positive numbers $\alpha_n \to 0$ such that $\|R_{\alpha_n}\| \to \infty$. Assume the converse: there is a constant *c* such that for any $\alpha > 0$ it holds $\|R_\alpha\| < c$. Then for any $y \in \mathcal{R}(A)$ the following inequalities hold:

$$\|A^{-1}y\| \leq \|A^{-1}y - R_\alpha y\| + \|R_\alpha y\| \leq \|x - R_\alpha Ax\| + \|R_\alpha\|\|y\| \leq \|x - R_\alpha Ax\| + c\|y\|.$$

The first term on the right side tends to zero (the definition of the regularization strategy) and hence we obtain the contradiction with $A^{-1}$ is unbounded.

Another moment in the regularization is the fact that the right hand side in (1) may involve some noise. Suppose that $b_\delta \in V$ is a perturbation of *b*, $\|b - b_\delta\| \leq \delta$. We define

$$x_{\alpha,\delta} = R_\alpha b_\delta. \tag{2}$$

The vector (2) is called the *regularized solution to the perturbed equation of (1).*

Let $x^* \in U$ be the exact solution to (1). Now we derive the fundamental estimate for regularization strategy:

$$\|x^* - x_{\alpha,\delta}\| \leq \|x^* - R_\alpha b\| + \|R_\alpha b - R_\alpha b_\delta\| \leq \|x^* - R_\alpha Ax^*\| + \|R_\alpha\|\delta. \tag{3}$$

As $\alpha \to 0+$ the first term on the right side in (3) tends to zero (*regularization effect*) whereas the second one grows to infinity (*ill-posedness efect*). We observe two competing effects which enter (3). These effects force us to make a trade off between accuracy and stability. The natural question is how to choose the value of parameter $\alpha$ when $\delta$ and $b_\delta$ are given. We briefly comment this within the discussion of Tikhonov regularization method.

Notice that the exact solution $x^*$ need not exist. On the other hand the regularized solution (2) always exists. If the classical solution does not exist, usually it can be replaced by so called generalized solution which can be regarded in different ways. The classical approach to the notion of the generalized solution is described in the following section.

## 1.2 Generalized solution in the sense of the least squares

We will suppose from now on that in equation (1) *U* and *V* are Hilbert spaces. If $b \notin \mathcal{R}(A)$, then there is no (classical) solution to (1). In that case we try to find $x_0 \in U$ for which $Ax_0$ is the closest element to *b*:

$$\|Ax_0 - b\| = \inf_{x \in U} \|Ax - b\|. \tag{4}$$

If such a vector $x_0$ exists we call it the *generalized solution to the equation (1) in the sense of least squares* (GSLS). The existence of GSLS is assured if $\mathcal{R}(A)$ is closed subspace of *V* (e.g. if $\mathcal{R}(A)$ is of finite dimension). If it is not the case the GSLS need not exist. If $\mathcal{R}(A)$ is a dense subspace of $V$ different from $V$ there exists $b \in \mathcal{R}(A)$ such that (1) does not have the GSLS.

Let $P: V \to V$ be the orthogonal projection onto $\overline{\mathcal{R}(A)}$ (the closure of the range of $A$ in $V$). Then the following statements are equivalent:

i. $x_0 \in U$ is the GSLS to (1).

ii. $Ax_0 = Px_0$.

iii. $A^*Ax_0 = A^*b$. (5)

The equation (5) is known as the *normal form of the equation (1).*

In general case the GSLS (if it exists) is not unique. The set of all GSLS´s is a linear manifold $\mathcal{G} = \left\{x_0 \in U: \|Ax_0 - b\| = \inf_{x\in U}\|Ax - b\|\right\} = x_0 + \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the kernel of the operator $A$. To reach the uniqueness we have to impose some additional requirement. We usually require that GSLS shall have the least norm. If $\mathcal{G} \neq \emptyset$ then there exists unique element $\tilde{x} \in \mathcal{G}$, so called the *best approximate solution to (1)*, such that $\|\tilde{x}\| = \inf_{x\in\mathcal{G}}\|x\|$.

In this context there is introduced the notion of the *generalized Moore-Penrose pseudoinverse* $A^\dagger: \mathcal{D} \to U$ defined on $\mathcal{D} = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$ mapping $b \in \mathcal{D}$ on the best approximate solution $\tilde{x} \in U$. The domain $\mathcal{D}$ of the pseudoinverse $A^\dagger$ is a dense subspace of $V$ equal to $V$ if and only if $\mathcal{R}(A)$ is a closed subspace of $V$. $A^\dagger$ is a linear operator which is closed once $A$ is closed and this is true if $A$ is bounded. As the consequence of the closed graph theorem we obtain that $A^\dagger$ is bounded if and only if $\mathcal{R}(A)$ is a closed subspace of $V$.

## 2 The characteristics and properties of the stochastic process

The *stochastic process* is a mapping

$$\boldsymbol{X}: \mathcal{T} \to L_2(\Omega, \pi). \tag{6}$$

The domain of the mapping (6), the set $\mathcal{T}$, is so called time domain. This set is usually the set of all integers (discrete time) or some interval in the real line (continuous time). Here it will be the former case and we will write $\boldsymbol{X} = (X_t)$ and talk about stochastic sequence. The values of (6) are the functions that are square integrable on a measurable space $\Omega$ with a probability measure $\pi$. These functions are called random variables. The framework of the space $L_2(\Omega, \pi)$ is suitable since the mean and variance of its elements (random variables) are finite. In particular, $L_2(\Omega, \pi)$ is the Hilbert space with the norm derived from the inner product $E(XY) = \int_{\Omega} XY d\pi$ for any two random variables $X, Y \in L_2(\Omega, \pi)$, the mean is $EX = \int_{\Omega} X d\pi$ and the variance is $DX = EX^2 - E^2X$.

The main characteristic of the stochastic process are the function of means $\mu_t = EX_t$ , the function of variances $\sigma_t = DX_t$ and the covariance function $C(t,s) = cov(X_t, X_s)$ respectively. An example of the stochastic sequence is the constant one $\boldsymbol{E} = (1)$ with $EX = 1$ and the other characteristics zero.

## 2.1 Stationary process and white noise

The stochastic process is called stationary if it has the following three properties (Arlt, 1999):

i. All the random variables have the same finite mean $\mu_X = EX_t$.

ii. All the random variables have the same finite variance $\sigma_X^2 = DX_t = EX_t^2 - E^2X_t$.

iii. The covariance is dependent only on time distance of the two random variables $\gamma_k = cov(X_t, X_s) = cov(X_{t-k}, X_{s-k})$.

The *autocovariance function* $\gamma_k$ of the process (6) is even, so we regard it only for non-negative *k*. Obviously $\gamma_0 = \sigma_X^2$.

In the special case, when the mean is zero, variance is constant and the random variables within the process are uncorrelated (i.e. $\gamma_k = 0$ for $k > 0$) the process is called *white noise*. We denote it $\mathcal{E} = (\varepsilon_t)$. White noise is an orthogonal system in $L_2(\Omega, \pi)$.

## 2.2 General linear process

The *general linear process* is the process of the form

$$X_t = \mu + \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k}, \tag{7}$$

where $\mu$ is a scalar and $(\psi_k)$ is a given sequence of scalars (weights of the process), $\psi_0 = 1$. The convergence of the series in (7) is intended in the sense of the convergence in the square mean and it is the same as in norm topology of the space $L_2(\Omega, \pi)$. This convergence is equivalent to the stationarity of the process (7). The necessary and sufficient condition for the convergence of (7) in the given sense is $(\psi_k) \in \ell_2$ (the space of all square summable sequences).

Another condition

$$\sum_{k=0}^{\infty} |\psi_k| < \infty \tag{8}$$

is only sufficient for the convergence of (7). It follows from the fact, that (8) means the sequence of weights $(\psi_k)$ is in the space $\ell_1$ (the space of absolutely convergent series) and it holds $\ell_1 \subset \ell_2$ (Horský, 2013).

## 3 Lag operator

The lag operator (backward shift operator) is intended to simplify formal writings in the context of difference equations. On the other hand it is a linear operator and thus it is a subject to be studied by means of the functional analysis. This operator will be regarded in two different normed linear spaces. Both spaces perform a suitable framework from relevant point of views.

### 3.1 The settings of the convergence structures

We will consider stochastic process in a slightly distinct form than in (6). The stochastic process will be a sequence of random variables

$$\boldsymbol{X} = (X_t, X_{t-1}, X_{t-2}, \dots) \tag{9}$$

for any $t \in \mathcal{T}$, where $X_{t-k} \in L_2(\Omega, \pi)$, $k \geq 0$. The sequence (9) will be considered as an element in two different spaces. The first one is $\ell_\infty(L_2)$, the space of all bounded stochastic sequences, with the norm

$$\|\boldsymbol{X}\|_\infty = \sup_{k \geq 0} \sqrt{EX_{t-k}^2}. \tag{10}$$

The space $\ell_\infty(L_2)$ with the norm (10) is the Banach space (like classical $\ell_\infty$, the space of all bounded scalar sequences.) The stationary sequences are contained in $\ell_\infty(L_2)$.

The second space is $\ell_2(L_2)$, the space of all square summable stochastic sequences which is the Hilbert space with respect to the norm derived from the inner product $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \sum_{k=0}^{\infty} E(X_{t-k} Y_{t-k})$, i.e.

$$\|\boldsymbol{X}\|_2 = \left( \sum_{k=0}^{\infty} E(X_{t-k}^2) \right)^{1/2}. \tag{11}$$

Even if this space has more comfortable structure than the first one (due to it is the Hilbert space), unfortunately it contains no stationary or constant stochastic sequence, unless the trivial (zero) sequence. It means that there is no white noise in it. This deficiency can be overcome if we adopt slightly modified notions of stationarity and white noise respectively. We allow the process to be stationary as far as one wishes then the norms (variances) has to fall to zero.

By the symbol $\mathcal{M}$ we denote the space $\ell_\infty(L_2)$ or $\ell_2(L_2)$ respectively. Then we define *the lag operator* (e.g. Dhrymes 1980) as a mapping

$$B: \mathcal{M} \to \mathcal{M}, \qquad B(X_t) = (X_{t-1}). \tag{12}$$

The norm of the operator (12) in the case of the space $\mathcal{M} = \ell_\infty(L_2)$ is $\|B\| = \sup_{\|\boldsymbol{X}\|_\infty = 1} \|B\boldsymbol{X}\|_\infty = 1$ and for any non-negative $k$ it holds that $\|B^k\| = 1$. The Banach algebra $\mathcal{L}(\ell_\infty(L_2))$ of all bounded linear operators on the space $\ell_\infty(L_2)$ contains polynomials in $B$. We may express any process (7) in the form

$$\boldsymbol{X} = \psi(B)\mathcal{E}, \tag{13}$$

where $\mathcal{E} = (\varepsilon_{t-k})$ is a white noise and

$$\psi(B) = \sum_{k=0}^{\infty} \psi_k B^k. \tag{14}$$

The series on the right side in (14) is convergent in the Banach space $\mathcal{L}(\ell_\infty(L_2))$, i.e. in the operator norm, if (8) holds. In fact,

$$\|\psi(B)\| \leq \sum_{k=0}^{\infty} |\psi_k| \|B^k\| = \sum_{k=0}^{\infty} |\psi_k|.$$

The operator (14) is called the *linear filter.* It transforms a white noise to a general linear process, see (13).

Example 3.1 The linear filter is called the *geometric lag operator*, if $\psi_k = \lambda^k$, $\lambda$ is a fixed scalar. The condition (8) is satisfied if and only if $|\lambda| < 1$. If $\lambda = 1$, the series (14) is divergent. In this case its partial sums define the divergent process called the *random walk.*

Example 3.2 An important class of stochastic processes is described by a stochastic difference equation (Arlt, 1999)

$$\Phi(B)\boldsymbol{X} = \Theta(B)\mathcal{E}, \tag{15}$$

where $\Phi(B)$ and $\Theta(B)$ are polynomials (in $B$) of the order $p$ and $q$, respectively. They are the well-known ARMA(*p,q*) processes if $\Phi(z) \neq 0$ for any $|z| \leq 1$. In such case the equation (15) has a unique solution and this solution is a stationary sequence. In terms of the previous chapter (15) is well-posed. However if $\Phi(1) = 0$, the equation is ill-posed as we explain in the next section. It means we encounter a nonstationary process. The process is denoted as ARIMA(*p,d,q*), where $d > 0$ is the multiplicity of the unit as a root of the $\Phi(z)$. We can watch of course only a finite part of this process and it remains a question why such a sequence (in infinite dimension) exists in some sense.

The norm of the lag operator (12) as an element of $\mathcal{L}(\ell_2(L_2))$ is equal to 1 as well as in $\mathcal{L}(\ell_\infty(L_2))$.

### 3.2 Spectral properties of the lag operator

The analysis of the spectrum of the lag operator starts with the fact that $\|B\| = 1$. It means that the spectrum of the operator $B$ is contained inside the unit circle $K(0) = \{\lambda \in \mathbb{C}: |\lambda| \leq 1\}$. In fact, this spectrum is the whole circle $K(0)$ since the spectrum of linear bounded operator is a non-empty compact set in the complex plane. The structure of this spectrum is analyzed in Horský, 2013. Main results are that the sequence $E_\lambda = (1, \lambda, \lambda^2, \dots)$ is for any $|\lambda| < 1$ an eigenvector of (12). The boundary of the unit circle is also in the point spectrum (Taylor, 1974) for only $B \in \mathcal{L}(\ell_\infty(L_2))$. If $B \in \mathcal{L}(\ell_2(L_2))$ and $|\lambda| = 1$ the operator $(\lambda I - B)^{-1}$ exists but it is not bounded and it may not be defined on the whole space $\ell_2(L_2)$ or its closed subspace. In fact the space $\mathcal{R}(\lambda I - B)$ is a proper subspace of $\ell_2(L_2)$ which is dense in $\ell_2(L_2)$. For example if we take $\lambda = 1$ the corresponding eigenvector in $\ell_\infty(L_2)$ is $\boldsymbol{E} = (1)$, however $\boldsymbol{E} \notin \ell_2(L_2)$.

The spectral analysis of the lag operator (12) provides the answers to the problem of solving the equation (15) with $\Phi(B) = \Delta = I - B$. Since the unit is in spectrum of $B$, the equation (15) is an ill-posed problem. In the following chapter we will deal with the case of the model of the random walk and its regularization. We use the well-known Tikhonov regularization method.

## 4 The regularization of the random walk

We come back to the equation (1), where $A = \Delta = I - B$ is the difference operator and the right hand side is a white noise. Thus the equation (1) has the form

$$(I - B)X = \mathcal{E}. \tag{16}$$

We will suppose the equation (16) in the Hilbert space $U = V = \ell_2(L_2)$. The Tikhonov regularization method consists in the regularization of the equation (5) or (17), see below. Since the problem (16) is ill-posed the problem (5) so is. It follows from several non-trivial facts (Lukeš, 2012). The spectrum of the adjoint operator $B^*$ is $\sigma(B^*) = \{z \in \mathbb{C}: |z| \leq 1\}$. Next $A^* = I - B^*$ and this operator has an unbounded inverse the domain of which is an open and dense subspace of $\ell_2(L_2)$: $\mathcal{R}(A^*A) \subset \mathcal{R}(A^*) \subset \ell_2(L_2)$, $\overline{\mathcal{R}(A^*A)} = \overline{\mathcal{R}(A^*)} = \ell_2(L_2)$ and $\mathcal{N}(A^*A) = \mathcal{N}(A) = \{o\}$. Note that the spectrum of the operator $A = \Delta = I - B$ is only translated spectrum of $B$, i.e. $\sigma(A) = \{z \in \mathbb{C}: |z - 1| \leq 1\}$. It holds $\|A\| = 2$. Since $\|A^*A\| = \|A\|^2 = 4$ we obtain $\sigma(A^*A) = [0,4]$.

The simple way how to regularize the problem

$$A^*A\boldsymbol{X} = A^*\mathcal{E} \tag{17}$$

is to add some positive multiple of the identity. In this way we obtain an operator equation

$$(A^*A + \alpha I)\boldsymbol{X} = A^*\mathcal{E}, \tag{18}$$

where $\alpha > 0$. The operator $A^*A + \alpha I$ is bounded with range equal to $\ell_2(L_2)$. Hence its inverse is bounded and (18) is well-posed problem. The operators

$$R_\alpha = (A^*A + \alpha I)^{-1}A^* \tag{19}$$

form a regularization strategy for the equation (17). If (17) has a solution $\boldsymbol{X}^{(\mathbf{0})}$, then the unique solution to (18), say $\boldsymbol{X}_{\boldsymbol{\alpha}}^{(\mathbf{0})} = R_\alpha\mathcal{E}$, is its approximation. If we replace the white noise $\mathcal{E}$ in (18) by its perturbation $\mathcal{E}_\delta$, we obtain the regularized solution to (17) $\boldsymbol{X}_{\boldsymbol{\alpha},\boldsymbol{\delta}}^{(\mathbf{0})} = R_\alpha\mathcal{E}_\delta$, see (2). The unique solution of (18) is at the same time the unique minimum of the *Tikhonov's functional* $J_\alpha(\boldsymbol{X}) = \|A\boldsymbol{X} - \mathcal{E}\|_2^2 + \alpha\|\boldsymbol{X}\|_2^2$ or $J_{\alpha,\delta}(\boldsymbol{X}) = \|A\boldsymbol{X} - \mathcal{E}_\delta\|_2^2 + \alpha\|\boldsymbol{X}\|_2^2$ respectively for perturbed form of the equation (18) (replace $\mathcal{E}$ by $\mathcal{E}_\delta$ in (18)).

It can be seen with the help of the polar decomposition and the Riesz functional calculus (Lukeš, 2012) that for the operators (19) holds

$$\|R_\alpha\| \leq \frac{1}{2\sqrt{\alpha}}. \tag{20}$$

It gives the estimate for the ill-posedness effect in (3). As to the regularization effect we will suppose that the exact solution to (17) satisfies $\boldsymbol{X}^{(\mathbf{0})} = A^*\boldsymbol{Z}$ for some $\boldsymbol{Z} \in \ell_2(L_2)$, in other words $\boldsymbol{X}^{(\mathbf{0})} \in \mathcal{R}(A^*)$, which can be interpreted as an assumption of certain smoothness of this solution. It can be easily verified (by the help of the estimate (20)) that

$$\left\|R_\alpha A\boldsymbol{X}^{(\mathbf{0})} - \boldsymbol{X}^{(\mathbf{0})}\right\| \leq \alpha\|R_\alpha\|\|\boldsymbol{Z}\| \leq \frac{\sqrt{\alpha}}{2}\|\boldsymbol{Z}\|. \tag{21}$$

Finally, if we substitute the estimates (20) and (21) to (2) we obtain the fundamental estimation (3) for Tikhonov regularization applied on the random walk in the sense of least squares (i.e. (18)):

$$\left\|\boldsymbol{X}^{(\mathbf{0})} - \boldsymbol{X}^{(\mathbf{0})}_{\alpha,\delta}\right\| \leq \left\|(A^*)^{-1}\boldsymbol{X}^{(\mathbf{0})}\right\|\frac{\sqrt{\alpha}}{2} + \frac{\delta}{2\sqrt{\alpha}}. \tag{22}$$

The first term in (22) reflects the regularization effect whereas the second one the ill-posedness effect. We have to ballance the parameters $\alpha$ and $\delta$ in such a way that $\frac{\delta}{2\sqrt{\alpha}} \to 0$ for $\alpha \searrow 0$ and $\delta \searrow 0$. One of the well-known strategy for the choice of $\alpha$ is the Morozov´s discrepancy principle (Nair, 2009).

## Conclusion

The polynomial operators in the equation (15) may have a unit root. It causes either non-stationarity or non-invertibility of this model. In essential form the unit root is contained in the model (16). In such case the equation (15) is an ill-posed problem. Then it has to be solved by a regularization method if we require to obtain some reasonable solution to this problem.

As the spectral analysis of the lag operator in $\ell_2(L_2)$ shows it remains to prove the existence of the exact solution (in some sense) to the original problem (16) since the range of the difference is not closed and not equal to $\ell_2(L_2)$; it is only dense subspace of $\ell_2(L_2)$.

The Tikhonov regularization method raise again the question about the overdifferencing of time series since the transformation of the equation (16) to its normal form contains in fact the second difference of the process.

## References

Arlt, J. (1999): Moderní metody modelování ekonomických časových řad. Ekopress, Praha.

Bell, W. (1987): A Note on Overdifferencing and the Equivalence of Seasonal Series Models with Monthly Means and Models with (0,0,1)12 Seasonal Parts when Theta=1, Journal of Business and Economic Statistics, Vol. 5, 383-387.

Dhrymes, P.J. (1980): Distributed Lags. Problems of Estimation and Formulation. North Holland, Amsterdam.

Hoderlein, S.; Holzmann, H. (2011): Demand Analysis as an Ill-Posed Inverse Problem with semiparametric Specification. Econometric Theory Vol. 27, Issue 3, 609-638

Horowitz, J.L. (2014): Ill-Posed Inverse Problems in Economics. Annual Review of Economics, Vol.6, 21-51.

Horský, R. (2013): The Lag Operator and Its Spectral Properties. Mathematical Methods in Economics, Proceedings Part I, 285-290.

Lu, S.; Mathe, P. (2014): Discrepancy based model selection in statistical inverse problems. Journal of complexity, Vol. 30, 290-308.

Lukeš, J. (2012): Zápisky z funkcionální analýzy. Univerzita Karlova v Praze, Nakladatelství Karolínum.

Nair, M.T. (2009): Linear Operator Equations, Approximation and Regularization. World Scientific, ISBN: 978-981-283-565-9.

Sanchez, I. (2002): Efficient forecasting in nearly non-stationary processes. Journal of Forecasting, Vol. 21, 1-26.

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