MODIFICATION OF THE CLASSICAL L-MOMENTS: TL-MOMENTS

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Abstract

Application of the method of moments for the parametric distribution is common in the construction of a suitable parametric distribution. However, moment method of parameter estimation does not produce good results. An alternative approach when constructing an appropriate parametric distribution for the considered data file is to use the so-called order statistics. This paper deals with the use of order statistics as the methods of L-moments and TL-moments of parameter estimation. L-moments have some theoretical advantages over conventional moments. L-moments have been introduced as a robust alternative to classical moments of probability distributions. However, L-moments and their estimations lack some robust features that belong to the TL-moments. TL-moments represent an alternative robust version of L-moments, which are called trimmed L-moments. This paper deals with the use of L-moments and TL-moments in the construction of models of wage distribution. Three-parametric lognormal curves represent the basic theoretical distribution whose parameters were simultaneously estimated by three methods of point parameter estimation and accuracy of these methods was then evaluated. There are method of TL-moments, method of L-moments and maximum likelihood method in combination with Cohen's method. A total of 328 wage distribution has been the subject of research.

Key words: Order statistics, probability density function, distribution function, quantile function, model of wage distribution

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1 L-Moments

Moments and cumulants are traditionally used to characterize the probability distribution or the observed data set in statistics. It is sometimes difficult to determine exactly what information about the shape of the distribution is expressed by its moments of third and higher order. Especially in the case of a small sample, numerical values of sample moments can be very different from the values of theoretical moments of the probability distribution from which the random sample comes. Particularly in the case of small samples, parameter estimations of the probability distribution obtained using the moment method are often markedly less accurate than estimates obtained using other methods, such as maximum likelihood method.

An alternative approach is to use the order statistics. Let X be a random variable having a distribution with distribution function $F(x)$ and with quantile function $x(F)$, and let $X_1, X_2, \ldots, X_n$ is a random sample of sample size n from this distribution. Then $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ are the order statistics of random sample of sample size n, which comes from the distribution of random variable X.

L-moments are analogous to conventional moments and are estimated based on linear combinations of order statistics, i.e. L-statistics. L-moments are an alternative system describing the shape of the probability distribution.

L-moments present the basis for a general theory, which includes the characterization and description of the theoretical probability distribution, characterization and description of the obtained sample data sets, parameter estimation of theoretical probability distribution and hypothesis testing of parameter values for the theoretical probability distribution. The theory of L-moments includes such established procedures such as the use of order statistics and Gini's middle difference and leads to some promising innovations in the area of measuring skewness and kurtosis of the distribution and provides relatively new methods of parameter estimation for individual distribution. L-moments can be defined for any random variable whose expected value exists. The main advantage of the L-moments than conventional moments consists in the fact that L-moments can be estimated on the basis of linear functions of the data and are more resistant to the influence of sample variation. Compared to conventional moments, L-moments are more robust to the existence of outliers in the data and allow better conclusions obtained on the basis of small samples for basic probability distribution. L-moments often bring even more efficient parameter estimations of parametric distribution than the estimations obtained using maximum likelihood method, especially for small samples. Theoretical advantages of L-moments over conventional moments lie in the ability to characterize a wider range of distribution and in greater resistance to the presence of outliers in the data when estimating from the sample. Compared with conventional moments, experience also shows that L-moments are less prone to bias estimation and approximation by asymptotic normal distribution is more accurate in finite samples.

1.1 L-Moments of Probability Distribution

Let X be a continuous random variable that has a distribution with distribution function $F(x)$ and with quantile function $x(F)$. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ are the order statistics of random sample of sample size n, which comes from the distribution of random variable X. L-moment of the r-th order of random variable X is defined

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r-j:r}), \quad r = 1, 2, \ldots . \tag{1}$$

Expected value of the r-th order statistic of random sample of sample size n has the form

$$E(X_{r:n}) = \frac{n!}{(r-1)! \cdot (n-r)!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-1} \cdot [1 - F(x)]^{n-r} \mathrm{d}\, F(x). \tag{2}$$

If we substitute equation (2) into equation (1), we obtain after adjustments

$$\lambda_r = \int_0^1 x(F) \cdot P_{r-1}^*[F(x)] \,\mathrm{d}\, F(x), \quad r = 1, 2, \ldots, \tag{3}$$

where

$$P_r^*[F(x)] = \sum_{j=0}^{r} p_{r,j}^* \cdot [F(x)]^j \quad \text{a} \quad p_{r,j}^* = (-1)^{r-j} \cdot \binom{r}{j} \cdot \binom{r+j}{j}, \tag{4}$$

and $P_r^*[F(x)]$ represents the *r*-th shifted Legendre polynomial. We also obtain substituting (2) into equation (1)

$$\lambda_r = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{r!}{(r-j-1)! \cdot j!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r-j-1} \cdot [1-F(x)]^j \,\mathrm{d}\, F(x), \quad r = 1, 2, \ldots. \tag{5}$$

The letter “L” in the name of “L-moments” stresses that the *r*-th L-moment λ_r is a linear function of the expected value of certain linear combination of order statistics. Own estimation of the *r*-th L-moment λ_r based on the obtained data sample is then linear combination of ordered sample values, i.e. L-statistics. The first four L-moments of the probability distribution in now defined

$$\lambda_1 = E(X_{1:1}) = \int_0^1 x(F) \,\mathrm{d}\, F(x), \tag{6}$$

$$\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) = \int_0^1 x(F) \cdot [2F(x) - 1] \,\mathrm{d}\, F(x), \tag{7}$$

$$\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) = \int_0^1 x(F) \cdot \{6[F(x)]^2 - 6F(x) + 1\} \,\mathrm{d}\, F(x), \tag{8}$$

$$\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) = \int_0^1 x(F) \cdot \{20[F(x)]^3 - 30[F(x)]^2 + 12[F(x)] - 1\} \,\mathrm{d}\, F(x). \tag{9}$$

The probability distribution can be specified by its L-moments, even if some of its conventional moments do not exist, but the opposite is not true. It can be proved that the first L-moment λ_1 is the level characteristic of the probability distribution, the second L-moment λ_2 is the variability characteristic, of a random variable *X*. It is convenient to standardize the higher L-moments λ_r, $r \geq 3$, to be independent on specific units of the random variable *X*. The ratio of L-moments of the *r*-th order of random variable *X* is defined

$$\tau_r = \frac{\lambda_r}{\lambda_2}, \quad r = 3, 4, \dots . \tag{10}$$

It is also possible to define such a function of L-moments, which is analogous to the classical coefficient of variation, i.e. the so-called L-coefficient of variation.

$$\tau = \frac{\lambda_2}{\lambda_1}. \tag{11}$$

The ratio of L-moments τ_3 is the skewness characteristic and the ratio of L-moments τ_4 is the kurtosis characteristic of the corresponding probability distribution. Main properties of the probability distribution are summarized very well by the following four characteristics: L-location λ_1, L-variation λ_2, L-skewness τ_3 and L-kurtosis τ_4. L-moments λ_1 and λ_2, L-coefficient of variation τ and ratios of L-moments τ_3 and τ_4 are the most useful measurements for characterizing the probability distribution. Their most important features are: the existence (if the expected value of the distribution exists, then all L-moments of the distribution exit, too) and uniqueness (if the expected value of the distribution exists, then L-moments define only one distribution, i.e. no two distributions have the same L-moments).

1.2 Sample L-Moments

We usually estimate L-moments using random sample, which is taken from an unknown distribution. Since the *r*-th L-moment λ_r is a function of the expected values of order statistics of random sample of sample size *r*, it is natural to estimate it using the so-called U-statistic, i.e. the corresponding function of sample order statistics (averaged over partial subsets of sample size *r*, which can be formed from the obtained random sample of sample size *n*).

Let $x_1, x_2, \ldots, x_n$ is a sample and $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ is an ordered sample. Then the *r*-th sample L-moment can be written as

$$l_r = \binom{n}{r}^{-1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} \cdots \sum \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot x_{i_{r-j}:n}, \quad r = 1, 2, \ldots, n. \tag{12}$$

Hence the first four sample L-moments have the form

$$l_1 = \frac{1}{n} \cdot \sum_i x_i, \tag{13}$$

$$l_2 = \frac{1}{2} \cdot \binom{n}{2}^{-1} \cdot \sum_{i>j} \sum (x_{i:n} - x_{j:n}), \tag{14}$$

$$l_3 = \frac{1}{3} \cdot \binom{n}{3}^{-1} \cdot \sum_{i>j>k} \sum \sum (x_{i:n} - 2x_{j:n} + x_{k:n}), \tag{15}$$

$$l_4 = \frac{1}{4} \cdot \binom{n}{4}^{-1} \cdot \sum_{i>j>k>l} \sum \sum \sum (x_{i:n} - 3x_{j:n} + 3x_{k:n} - x_{l:n}). \tag{16}$$

U-statistics are widely used especially in nonparametric statistics. Their positive features are: the absence of bias, asymptotic normality and some slight resistance due to the influence of outliers.

When calculating the *r*-th sample L-moment it is not necessary to repeat the calculation across all partial subsets of sample size *r*, but this statistic can be expressed directly as linear combination of order statistics of random sample of sample size *n*. If we consider the estimation of $E(X_{r:r})$, which is taken using U-statistics, this estimate can be written as $r \cdot b_{r-1}$, where

$$b_r = \frac{1}{n} \cdot \binom{n-1}{r}^{-1} \cdot \sum_{j=r+1}^{n} \binom{j-1}{r} \cdot x_{j:n}, \tag{17}$$

specifically

$$b_0 = \frac{1}{n} \cdot \sum_{j=1}^{n} x_{j:n}, \tag{18}$$

$$b_1 = \frac{1}{n} \cdot \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} \cdot x_{j:n}, \tag{19}$$

$$b_2 = \frac{1}{n} \cdot \sum_{j=3}^{n} \frac{(j-1) \cdot (j-2)}{(n-1) \cdot (n-2)} \cdot x_{j:n}, \tag{20}$$

therefore generally

$$b_r = \frac{1}{n} \cdot \sum_{j=r+1}^{n} \frac{(j-1) \cdot (j-2) \cdot \ldots \cdot (j-r)}{(n-1) \cdot (n-2) \cdot \ldots \cdot (n-r)} \cdot x_{j:n}. \tag{21}$$

Therefore the firs sample L-moments can be written as

$$l_1 = b_0, \tag{22}$$

$$l_2 = 2b_1 - b_0, \tag{23}$$

$$l_3 = 6b_2 - 6b_1 + b_0, \tag{24}$$

$$l_4 = 20\,b_3 - 30\,b_2 + 12\,b_1 - b_0. \tag{25}$$

Thus, we can write universally

$$l_{r+1} = \sum_{k=0}^{r} p^{*}_{r,k} \cdot b_k, \quad r = 0, 1, \ldots, n-1, \tag{26}$$

where

$$p^{*}_{r,k} = (-1)^{r-k} \cdot \binom{r}{k} \cdot \binom{r+k}{k} = \frac{(-1)^{r-k} \cdot (r+k)!}{(k!)^2 \cdot (r-k)!}. \tag{27}$$

Application of sample L-moments is similar to the application of sample conventional moments. Sample L-moments summarize the basic properties of the sample distribution, which are the location (level), variability, skewness and kurtosis. Thus, sample L-moments estimate the corresponding properties of the probability distribution from which the sample comes and can be used in estimating the parameters of the relevant theoretical probability distribution. Under such applications, we often prefer the L-moments before conventional moments, since as a linear function of data, sample L-moments are less sensitive to the sample variability than conventional moments or to the size of errors in the case of existence of outliers. L-moments therefore lead to more accurate and robust estimations of the parameters or characteristics of a basic probability distribution. Sample L-moments were used already previously in the statistics, although not as a part of a unified theory. The first sample L-moment l_1 is a sample L-location (sample average), the second sample L-moment l_2 is a sample L-variability.

Natural estimation of the ratio of L-moments (10) is the sample ratio of L-moments

$$t_r = \frac{l_r}{l_2}, \quad r = 3, 4, \dots. \tag{28}$$

Hence t_3 is a sample L-skewness and t_4 is a sample L-kurtosis. Sample ratios of L-moments t_3 and t_4 can be used as characteristics of skewness and kurtosis of the sample data file. Gini's middle difference is related to sample L-moments, which has the form

$$G = \binom{n}{2}^{-1} \cdot \sum_{i>j}\sum (x_{i:n} - x_{j:n}), \tag{29}$$

and Gini's coefficient, which depends only on a single parameter σ in the case of two-parametric lognormal distribution, but it depends on the values of all three parameters in the case of three-parametric lognormal distribution.

2 TL-Moments

Alternative robust version of L-moments will be now presented. This robust modification of L-moments is called „trimmed L-moments“, and labeled „TL-moments“.

This is a relatively new category of moment characteristics of the probability distribution. There are the characteristics of the level, variability, skewness and kurtosis of probability distributions constructed using TL-moments that are robust extending of L-moments. L-moments alone were introduced as a robust alternative to classical moments of probability distributions. However, L-moments and their estimations lack some robust properties that belong to the TL-moments.

Sample TL-moments are linear combinations of sample order statistics, which assign zero weight to a predetermined number of sample outliers. Sample TL-moments are unbiased estimations of the corresponding TL-moments of probability distributions. Some theoretical and practical aspects of TL-moments are still under research or remain for future research. Efficiency of TL-statistics depends on the choice of α proportion, for example, the first sample TL-moments $l_1^{(0)}, l_1^{(1)}, l_1^{(2)}$ have the smallest variance (the highest efficiency) among other estimations from random samples from normal, logistic and double exponential distribution.

When constructing the TL-moments, the expected values of order statistics of random sample in the definition of L-moments of probability distributions are replaced by the expected values of order statistics of a larger random sample, where the sample size grows like this, so that it will correspond to the total size of modification, as shown below.

TL-moments have certain advantages over conventional L-moments and central moments. TL-moment of probability distribution may exist even if the corresponding L-moment or central moment of the probability distribution does not exist, as it is the case of Cauchy's distribution. Sample TL-moments are more resistant to existence of outliers in the data. The method of TL-moments is not intended to replace the existing robust methods, but rather as their supplement, especially in situations where we have outliers in the data.

2.1 TL-Moments of Probability Distribution

In this alternative robust modification of L-moments, the expected value $E(X_{r-j:r})$ is replaced by the expected value $E(X_{r+t_1-j:r+t_1+t_2})$. Thus, for each r we increase sample size of random sample from the original r to $r + t_1 + t_2$ and we work only with the expected values of these r treated order statistics $X_{t_1+1:r+t_1+t_2}, X_{t_1+2:r+t_1+t_2}, \ldots, X_{t_1+r:r+t_1+t_2}$ by trimming the t_1 smallest and the t_2 largest from the conceptual sample. This modification is called the r-th trimmed L-moment (TL-moment) and is marked $\lambda_r^{(t_1,t_2)}$. Thus, TL-moment of the r-th order of random variable X is defined

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r+t_1-j:r+t_1+t_2}), \quad r = 1, 2, \ldots. \tag{30}$$

It is apparent from equations (30) and (1) that the TL-moments simplify to L-moments, when $t_1 = t_2 = 0$. Although we can also consider applications, where the values of trimming are not equal, i.e. $t_1 \neq t_2$, we focus here only on symmetric case $t_1 = t_2 = t$. Then equation (30) can be rewritten

$$\lambda_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot E(X_{r+t-j:r+2t}), \quad r = 1, 2, \ldots. \tag{31}$$

Thus, form example, $\lambda_1^{(t)} = E(X_{1+t:1+2t})$ is the expected value of median from conceptual random sample of sample size $1 + 2t$. It is necessary here to note that $\lambda_1^{(t)}$ is equal to zero for distributions, which are symmetrical around zero.

First four TL-moments have the form for $t = 1$

$$\lambda_1^{(1)} = E(X_{2:3}), \tag{32}$$

$$\lambda_2^{(1)} = \frac{1}{2} E(X_{3:4} - X_{2:4}), \tag{33}$$

$$\lambda_3^{(1)} = \frac{1}{3} E(X_{4:5} - 2X_{3:5} + X_{2:5}), \tag{34}$$

$$\lambda_4^{(1)} = \frac{1}{4} E(X_{5:6} - 3X_{4:6} + 3X_{3:6} - X_{2:6}). \tag{35}$$

Note that the measures of location (level), variability, skewness and kurtosis of the probability distribution analogous to conventional L-moments (6)−(9) are based on $\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}$ a $\lambda_4^{(1)}$.

Expected value $E(X_{r:n})$ can be written using the formula (2). Using equation (2) we can re-express the right side of equation (31)

$$\lambda_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{(r+2t)!}{(r+t-j-1)! \cdot (t+j)!} \cdot \int_0^1 x(F) \cdot [F(x)]^{r+t-j-1} \cdot [1-F(x)]^{t+j} \,\mathrm{d}\,F(x),\ r = 1, 2, \ldots. \tag{36}$$

It is necessary to be noted here that $\lambda_r^{(0)} = \lambda_r$ is a normal the r-th L-moment without any trimming.

Expressions (32)-(35) for the first four TL-moments, where $t = 1$, can be written in an alternative manner

$$\lambda_1^{(1)} = 6 \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \,\mathrm{d}\,F(x), \tag{37}$$

$$\lambda_2^{(1)} = 6 \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot [2F(x) - 1] \,\mathrm{d}\,F(x), \tag{38}$$

$$\lambda_3^{(1)} = \frac{20}{3} \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot \{5[F(x)]^2 - 5F(x) + 1\} \,\mathrm{d}\,F(x), \tag{39}$$

$$\lambda_4^{(1)} = \frac{15}{2} \cdot \int_0^1 x(F) \cdot [F(x)] \cdot [1 - F(x)] \cdot \{14[F(x)]^3 - 21[F(x)]^2 + 9[F(x)] - 1] \, \mathrm{d}\, F(x). \qquad (40)$$

Distribution may be identified by its TL-moments, although some of its L-moments or conventional central moments do not exit; for example $\lambda_1^{(1)}$ (expected value of median of conceptual random sample of sample size three) exists for Cauchy's distribution, although the first L-moment λ_1 does not exist. TL-skewness $\tau_3^{(t)}$ and TL-kurtosis $\tau_4^{(t)}$ are defined analogously as L-skewness τ_3 and L-kurtosis τ_4

$$\tau_3^{(t)} = \frac{\lambda_3^{(t)}}{\lambda_2^{(t)}}, \qquad (41)$$

$$\tau_4^{(t)} = \frac{\lambda_4^{(t)}}{\lambda_2^{(t)}}. \qquad (42)$$

2.2 Sample TL-Moments

Let $x_1, x_2, \ldots, x_n$ is a sample and $x_{1:n} \le x_{2:n} \le \cdots \le x_{n:n}$ is an ordered sample. Expression

$$\hat{E}(X_{j+1:j+l+1}) = \frac{1}{\binom{n}{j+l+1}} \cdot \sum_{i=1}^{n} \binom{i-1}{j} \cdot \binom{n-i}{l} \cdot x_{i:n} \qquad (43)$$

is considered to be an unbiased estimation of expected value of the $(j + 1)$-th order statistic $X_{j+1:j+l+1}$ in conceptual random sample of sample size $(j + l + 1)$. Now we will assume that we replace the expression $E(X_{r+t-j:r+2t})$ by its unbiased estimation in the definition of the r-th TL-moment $\lambda_r^{(t)}$ in (31)

$$\hat{E}(X_{r+t-j:r+2t}) = \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \qquad (44)$$

which we gain by assigning $j \rightarrow r + t - j - 1$ a $l \rightarrow t + j$ in (43). Now we obtain the r-th sample TL-moment

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \hat{E}(X_{r+t-j:r+2t}), \quad r = 1, 2, ..., n-2t, \tag{45}$$

i.e.

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \cdot \frac{1}{\binom{n}{r+2t}} \cdot \sum_{i=1}^{n} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j} \cdot x_{i:n}, \quad r = 1, 2, ..., n-2t, \tag{46}$$

which is unbiased estimation of the r-th TL-moment $\lambda_r^{(t)}$. Note that for each $j = 0, 1, \ldots, r-1$, values $x_{i:n}$ in (46) are nonzero only for $r + t - j \leq i \leq n - t - j$ due to the combinatorial numbers. Simple adjustment of the equation (46) provides an alternative linear form

$$l_r^{(t)} = \frac{1}{r} \cdot \sum_{i=r+t}^{n-t} \left[\frac{\sum_{j=0}^{r-1} (-1)^j \cdot \binom{r-1}{j} \binom{i-1}{r+t-j-1} \cdot \binom{n-i}{t+j}}{\binom{n}{r+2t}} \right] \cdot x_{i:n}. \tag{47}$$

For example, we obtain for $r = 1$ for the first sample TL-moment

$$l_1^{(t)} = \sum_{i=t+1}^{n-t} w_{i:n}^{(t)} \cdot x_{i:n}, \tag{48}$$

where the weights are given by

$$w_{i:n}^{(t)} = \frac{\binom{i-1}{t} \cdot \binom{n-i}{t}}{\binom{n}{2t+1}}. \tag{49}$$

The above results can be used to estimate TL-skewness and TL-kurtosis by simple ratios

$$t_3^{(t)} = \frac{l_3^{(t)}}{l_2^{(t)}}, \tag{50}$$

$$t_4^{(t)} = \frac{l_4^{(t)}}{l_2^{(t)}}. \tag{51}$$

3 Appropriateness of the Model

It is also necessary to assess the suitability of constructed model or choose a model from several alternatives, which is made by some criterion, which can be a sum of absolute deviations of the observed and theoretical frequencies for all intervals

$$S = \sum_{i=1}^{k} \left| n_i - n\pi_i \right| \tag{52}$$

or known criterion χ^2

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - n\,\pi_i)^2}{n\,\pi_i}, \tag{53}$$

where n_i are the observed frequencies in individual intervals, π_i are the theoretical probabilities of membership of statistical unit into the i-th interval, n is the total sample size of corresponding statistical file, $n \cdot \pi_i$ are the theoretical frequencies in individual intervals, $i = 1, 2, ..., k$, and k is the number of intervals.

The question of the appropriateness of the given curve for model of the distribution of wage is not entirely conventional mathematical-statistical problem in which we test the null hypothesis “H_0: The sample comes from the supposed theoretical distribution” against the alternative hypothesis “H_1: non H_0”, because in goodness of fit tests in the case of wage distribution we meet frequently with the fact that we work with large sample sizes and therefore the tests would almost always lead to the rejection of the null hypothesis. This results not only from the fact that with such large sample sizes the power of the test is so high at the chosen significance level that the test uncovers all the slightest deviations of the actual wage distribution and a model, but it also results from the principle of construction of the test. But practically we are not interested in such small deviations, so only gross agreement of the model with reality is sufficient and we so called “borrow” the model (curve). Test criterion χ^2 can be used in that direction only tentatively. When evaluating the suitability of the model we proceed to a large extent subjective and we rely on experience and logical analysis.

4 Database

The database of the research consists in employees of the Czech Republic. There are a total set of all employees of the Czech Republic together and further the partial sets broken down by various demographic and socio-economic factors. The researched variable is the gross monthly wage in CZK (nominal wage). Data come from the official website of the Czech Statistical Office. The data was in the form of interval frequency distribution, since the individual data is not currently available. Researched period represents years 2003−2010. Employees of the Czech Republic were classified according to gender, job classification (CZ-ISCO), the classification of economic activities, age and educational attainment. Branch Classification of Economic Activities (OKEC) has been replaced by Classification of Economic Activities (CZ-NACE) during researched period. This fact therefore disrupts the continuity of the obtained time series during the analysis period. All calculations were made using the statistical program packages Statgraphics and SAS, spreadsheet Microsoft Excel.

5 Main Results

Table 1 presents parameter estimations obtained using the various three methods of point parameter estimation and the value of criterion *S* for the total wage distribution of the Czech Republic. This table describes approximately the research results of all 328 wage distribution. We obtained in total research that the method of TL-moments provided the most accurate results in almost all cases of wage distribution with minor exceptions, deviations occur mainly at both ends of the wage distribution due to the extreme open intervals of interval frequency distribution. In the results of Table 1 for total sets of wage distribution of the Czech Republic in 2003–2010 method of TL-moments always brings the most accurate results in terms of criterion *S*.

In terms of research of all 328 wage distribution, method of L-moments brought the second most accurate results in more than in half of the cases. Deviations occur again especially at both ends of the distribution. In the results of Table 1 method of L-moments brought the second most accurate results in terms of all total sets of wage distribution of the Czech Republic in 2003–2010.

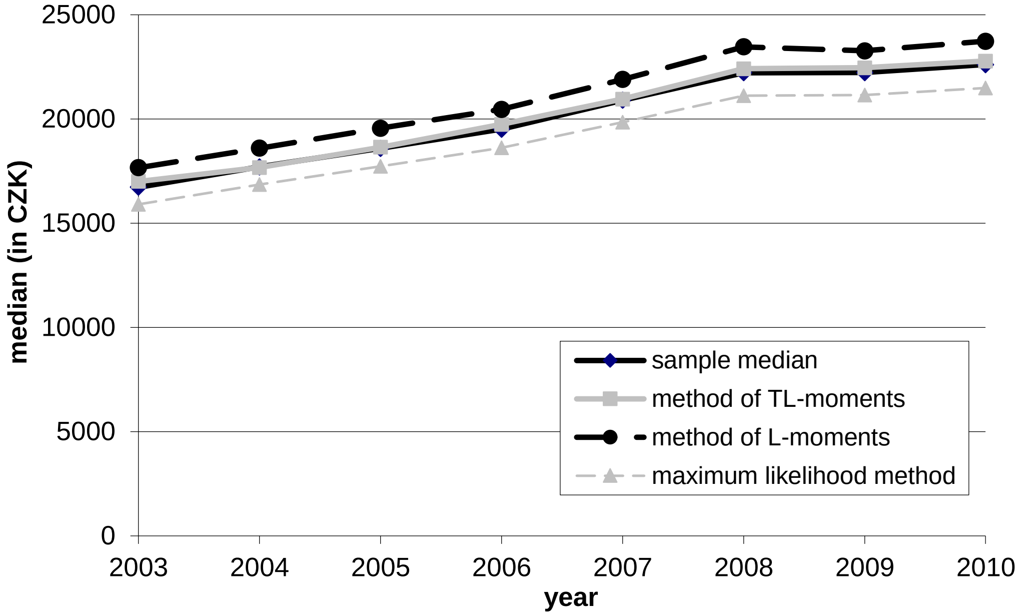
Overall, maximum likelihood method was the third in most cases in terms of accuracy of the results obtained (in all cases in Table 1). Figure 1 also gives some idea of the accuracy of the researched methods of point parameter estimation. This figure shows the development of the sample median of gross monthly wage for the total set of all employees of the Czech Republic together in the period 2003–2010 and the development of corresponding theoretical median of model three-parametric lognormal curves with parameters estimated by three various methods of point parameter estimation. We can observe from this figure that the curve characterizing the course of theoretical median of three-parametric lognormal distribution with parameters estimated using the method of TL-moments adheres the most to the curve showing the development of the sample median.

Tab. 1: Parameter estimations obtained using the various three methods of point parameter estimation and the value of criterion *S* for the total wage distribution of the Czech Republic

Year	Method								
	Method of TL-moments			Method of L-moments			Maximum likelihood method in combination with Cohen’s method		
	Parameter estimation			Parameter estimation			Parameter estimation		
	μ	σ^2	θ	μ	σ^2	θ	μ	σ^2	θ
2003	9.059 747	0.630 754	9,065.52	9.017 534	0.608 369	7,664.46	9.741 305	0.197 395	2.07
2004	9.215 324	0.581 251	8,552.10	9.241 235	0.507 676	6,541.16	9.780 008	0.232 406	0.22
2005	9.277 248	0.573 002	8,872.54	9.283 399	0.515 290	6,977.45	9.833 604	0.228 654	0.27
2006	9.313 803	0.577 726	9,382.66	9.283 883	0.543 225	7,868.21	9.890 594	0.210 672	0.59
2007	9.382 135	0.680 571	10,027.84	9.387 739	0.601 135	7,902.64	9.950 263	0.268 224	0.16
2008	9.438 936	0.688 668	10,898.39	9.423 053	0.624 340	8,754.64	10.017 433	0.264 124	0.19
2009	9.444 217	0.703 536	10,640.53	9.431 478	0.631 013	8,684.51	10.019 787	0.269 047	0.20
2010	9.482 060	0.681 258	10,616.80	9.453 027	0.621 057	8,746.20	10.033 810	0.269 895	0.20
	Criterion *S*			Criterion *S*			Criterion *S*		
2003		108,437.01			133,320.79			248,331.74	
2004		146,509.34			248,438.78			281,541.41	
2005		137,422.05			231,978.79			311,008.23	
2006		149,144.68			216,373.24			325,055.67	
2007		198,670.74			366,202.87			370,373.62	
2008		206,698.93			357,668.48			391,346.02	
2009		193,559.55			335,999.20			359,736.37	
2010		200,060.20			339,871.80			378,629.15	

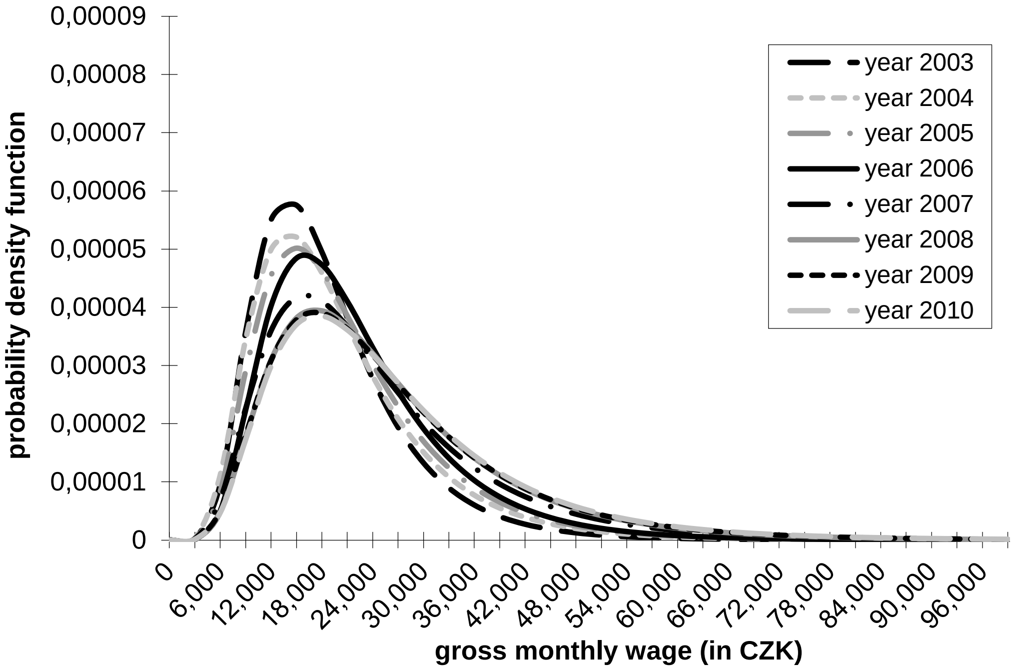
Source: Own research

Fig. 1. Development of sample and theoretical median of three-parametric lognormal curves with parameters estimated using three various methods of parameter estimation



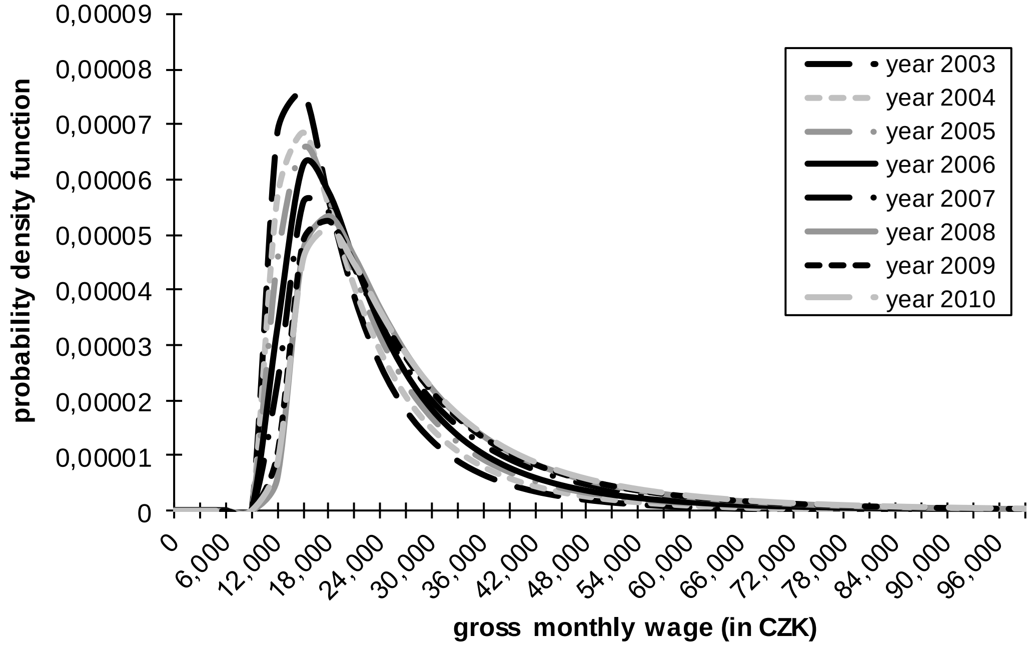
Source: Own research

Fig. 2: Development of probability density function of three-parametric lognormal curves with parameters estimated using the method of TL-moments



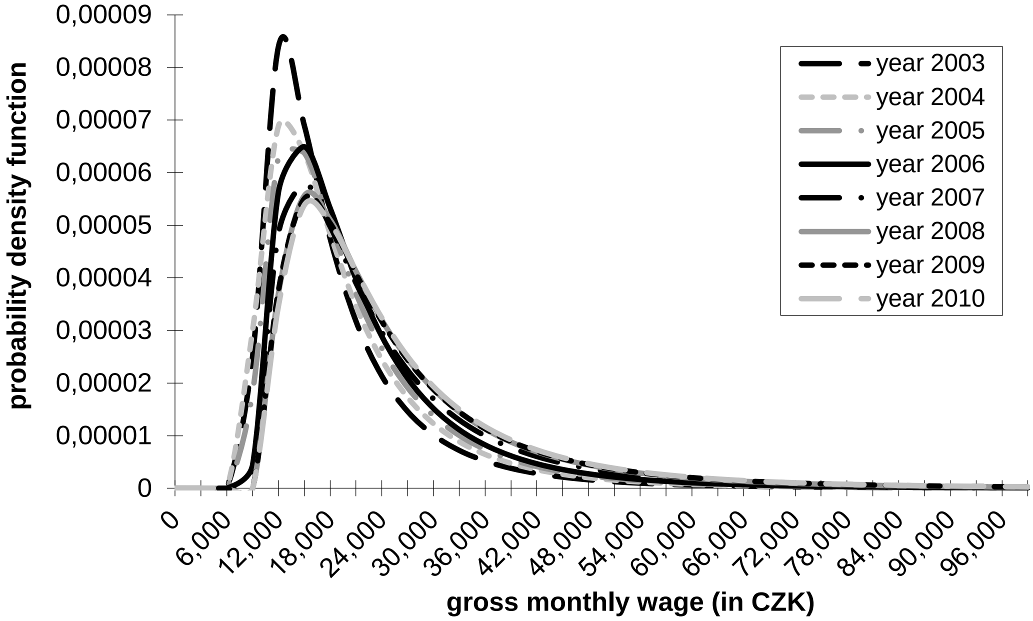
Source: Own research

Fig. 3: Development of probability density function of three-parametric lognormal curves with parameters estimated using the method of L-moments



Source: Own research

Fig. 4: Development of probability density function of three-parametric lognormal curves with parameters estimated using the maximum likelihood method



Source: Own research

The other two curves articulating the development of the theoretical median of three-parametric lognormal curves with parameters estimated by method of L-moments and by maximum likelihood method are relatively remote from the course of sample median of wage distribution.

Figures 2–4 represents the development of probability density function of three-parametric lognormal curves with parameters estimated using the method of TL-moments, method of L-moments and maximum likelihood method. This is again a development of model distributions of the total wage distribution of the Czech Republic for all employees of the Czech Republic together in the period 2003—2010. We can see that the shapes of the lognormal curves with parameters estimated using the method of L-moments and maximum likelihood method (Figures 3 and 4) are similar mutually and they are very different from the shape of three-parametric lognormal curves with parameters estimated by the method of TL-moments (Figure 2).

Conclusion

Alternative category of moment characteristics of probability distributions was introduced here. There are the characteristics in the form of L-moments and TL-moments. Accuracy of the methods of L-moments and TL-moments was compared with the accuracy of the maximum likelihood method using such criterion as the sum of all absolute deviations of the observed and theoretical frequencies for all intervals. Higher accuracy of the method of TL-moments due to the method of L-moments and to the maximum likelihood method was proved by studying of the set of 328 wage distribution. However, the advantages of the method of L-moments to the maximum likelihood method were demonstrated here, too. The values of χ^2 criterion were also calculated for each wage distribution, but this test led always to the rejection of the null hypothesis about the supposed shape of the distribution even at 1% significance level due to the large sample sizes, which are typical for wage distribution. The dependence the value of criterion χ^2 and the value of criterion of the sum of all absolute deviations of observed and theoretical frequencies on the sample size follows from the construction of the test.

A number of authors deals with the issue of the labor market and living standards of the population of the Czech Republic, see for example (Bartošová & Bártová, 2014), (Bartošová & Longford, 2014), (Pavelka & Löster, 2013), (Pivoňka & Löster, 2014) and (Šimpach & Pechrová, 2013). A number of authors study directly the problems of wages and incomes, see for example (Malá, 2014), (Marek, 2013), (Marek & Vrabec, 2013) and (Pavelka, Skála & Čadil, 2014). Such authors as (Löster, 2014), (Malá, 2013), (Malec & Malec, 2013) and (Šimpach, 2012) research some statistical methods, which can used for economic data.

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