ROBUST RECURSIVE ESTIMATION FOR FINANCIAL TIME SERIES

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Abstract

The generalized autoregressive conditional heteroscedasticity (GARCH) process is a particular modelling scheme, which is capable of forecasting the current level of volatility of financial time series. Recently, recursive estimation methods suitable for this class of stochastic processes have been introduced in the literature. They undoubtedly represent attractive alternatives to the standard non-recursive estimation procedures with many practical applications. It is truly advantageous to adopt numerically effective estimation techniques that can estimate and control such models in real time. However, abnormal observations (outliers) may occur in data. They may be caused by many reasons, e.g. by additive errors, measurement failures or management actions. Exceptional data points will influence the model estimation considerably if no specific action is taken. The aim of this contribution is to propose and examine a robust recursive estimation algorithm suitable for GARCH models. It seems to be useful for various financial time series, in particular for (high-frequency) financial returns contaminated by additive outliers. The introduced algorithm can be effective in the risk control and regulation when the prediction of volatility is the main concern since it distinguishes and corrects outlaid bursts of volatility. Real data examples are presented.

Key words: financial time series, GARCH, recursive estimation, robust methods, volatility

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Introduction

Financial time series (in particular returns of financial assets) typically exhibit significant kurtosis and volatility clustering. The assets are usually stocks or stock indices or currencies (Tsay, 2013). The GARCH models introduced by Engle (1982) and Bollerslev (1986) are commonly applied in order to model these typical properties with the aim to describe dynamics of conditional variances and forecast financial volatility. However, when fitted to real time series the standardized residuals of the estimated models have frequently excess kurtosis explainable by the presence of outliers which are not captured by the GARCH models, see e.g.

Carnero, Peña, and Ruiz (2012), Charles (2008) or Charles and Darné (2005). On the other hand, some authors argue that extreme observations are not outliers and they should be incorporated into the model, see e.g. Eraker, Johannes, and Polson (2003).

The parameters of the GARCH models are routinely estimated by the (conditional) maximum likelihood but they are rarely calibrated recursively. Nevertheless, parameter estimation performed using recursive algorithms is undoubtedly advantageous. To evaluate the parameter estimates at a time step, recursive estimation methods operate only with the current measurements and parameters estimated in previous steps (see e.g. Aknouche and Guerbyenne (2006) or Hendrych and Cipra (2018)). It is in sharp contrast to the non-recursive estimation where all data are collected at first and then the model is fitted. Therefore, recursive estimation techniques are effective in terms of memory storage and computational complexity. This efficiency can be employed just in the framework of (high-frequency) financial time series data. Alternatively, it is possible to adopt these methods to monitor or forecast volatility in real time, to evaluate risk measures (e.g. the Value at Risk or Expected Shortfall), to detect faults, to check model stability including detection of structural changes, etc. Moreover, due to the previous arguments, the recursive GARCH estimation should be resistant (robust) to outliers. The primary goal of this paper is to suggest a robust recursive algorithm which is effective enough in the context of GARCH models to estimate and forecast volatility of contaminated (high-frequency) financial data in real time.

Various methods of non-recursive estimation of GARCH parameters and volatility in presence of outliers consist either in: (*i*) identifying and correcting additive outliers (AO) or innovative outliers (IO) in (residual) time series (see e.g. Charles (2008) or Charles and Darné (2005)), (*ii*) robustifying classical statistical estimators of the type LS or ML to the form of M-estimators and similar robust versions (see e.g. Carnero et al. (2012)), or (*iii*) applying estimators with robust properties of the type LAD or MAD (see e.g. Bernholt, Fried, Gather, and Wegener (2006)). As robust recursive estimation of GARCH model is concerned, initially one should remind a close connection to robustification of Kalman filter, which is desirable including various engineering applications in the context of state space modeling with outliers (see e.g. Romera and Cipra (1995)).

This paper is organized as follows. Section 1 presents robustified self-weighted recursive estimation algorithm suitable for GARCH models. Section 2 describes an empirical study in which various estimation strategies leading to different volatility forecasts are compared real data applications.

1 Robust recursive estimation of GARCH models

The GARCH(p, q) process $\{y_t\}_{t\in Z}$ in financial applications is commonly defined as:

$$y_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 , \tag{1}$$

where $\{\varepsilon_t\}_{t\in Z}$ is a sequence of *iid* random variables with zero mean and unit variance, and ω, $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ are real parameters of the process. The first two conditional and unconditional moments can be simply calculated as:

$$\mathrm{E}(y_t | \Im_t) = 0, \quad \mathrm{E}(y_t) = 0, \quad \mathrm{var}(y_t | \Im_t) = \sigma_t^2, \quad \mathrm{var}(y_t) = \mathrm{E}(\sigma_t^2) , \tag{2}$$

where $\Im_t$ denotes the smallest σ-algebra with respect to which y_s is measurable for all $s \le t$. Sufficient conditions for σ_t^2 being positive are $\omega > 0$, $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \ge 0$. If $\beta_1 = \ldots = \beta_q = 0$, the model is reduced to the ARCH(p) case. Additionally, sufficient conditions for y_t being (weakly) stationary are $\omega > 0$, $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \ge 0$ and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$.

The one-step ahead prediction of σ_t^2 is expressed as:

$$\sigma_{t+1|t}^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t+1-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t+1-j}^2 . \tag{3}$$

The GARCH models are routinely estimated by the non-recursive conditional maximum likelihood method with normal distribution being usually preferred since the corresponding estimates stay consistent (even if they might be inefficient in the case of non-normally distributed innovations ε_t in (1)).

Hendrych and Cipra (2018) proposed the recursive scheme for estimating the parameters of the GARCH model (1). The algorithm was derived by applying the general recursive prediction error method (Ljung & Söderström, 1983). Principally, the negative conditional log-likelihood criterion corresponding to the GARCH process is recursively minimized (when assuming normally distributed innovations ε_t in (1)). In many instances, this approach may be truly advantageous. For example, it is possible to monitor or predict volatility on-line in the high-frequency financial data context. Recursive estimation methods are also effective in terms of memory storage and computational complexity since the current parameter estimates are evaluated using the previous estimates and actual measurements. Moreover, they can be used to detect structural model changes.

Using GARCH models, it is necessary to be concerned about outliers that may occur in data (see also Introduction and Section 2 on real data applications). Outliers can be caused by many reasons, e.g. by additive innovations, measurement failures, operational risk problems, management decisions, etc. They can influence the estimation and prediction in the applied model considerably if no specific action is taken. Therefore, if such defects are expected in the data set, one should modify the estimation algorithms to make them more robust. The outliers tend to appear as spikes in the sequence of standardized residuals which obviously result in large contributions to the loss function. There exist various ways how to robustify recursive estimation algorithms (refer to Introduction). In this contribution, a simple way of handling outliers is applied based on testing a measurement at each time *t*. If it is large compared with a given limit, it is indicated as erroneous and substituted immediately by another value (consult e.g. Romera and Cipra (1995) and others). According to simulations, this strategy seems to be efficient for additive outliers (AD) mainly.

Under the previous arguments, the recursive estimator introduced by Hendrych and Cipra (2018) can be robustified to the following form:

$$\hat{\boldsymbol{\theta}}_t^{rob} = \hat{\boldsymbol{\theta}}_{t-1}^{rob} + \frac{\hat{\boldsymbol{P}}_{t-1}^{rob}\hat{\boldsymbol{\psi}}_t^{rob}\left[(\hat{y}_t^{rob})^2 - (\hat{\boldsymbol{\varphi}}_t^{rob})'\hat{\boldsymbol{\theta}}_{t-1}^{rob}\right]}{\lambda_t\left[(\hat{\boldsymbol{\varphi}}_t^{rob})'\hat{\boldsymbol{\theta}}_{t-1}^{rob}\right]^2 + (\hat{\boldsymbol{\psi}}_t^{rob})'\boldsymbol{P}_{t-1}^{rob}\hat{\boldsymbol{\psi}}_t^{rob}},$$

$$\hat{\boldsymbol{P}}_t = \frac{1}{\lambda_t}\left\{\hat{\boldsymbol{P}}_{t-1}^{rob} - \frac{\hat{\boldsymbol{P}}_{t-1}^{rob}\hat{\boldsymbol{\psi}}_t^{rob}(\hat{\boldsymbol{\psi}}_t^{rob})'\hat{\boldsymbol{P}}_{t-1}^{rob}}{\lambda_t\left[(\hat{\boldsymbol{\varphi}}_t^{rob})'\hat{\boldsymbol{\theta}}_{t-1}^{rob}\right]^2 + (\hat{\boldsymbol{\psi}}_t^{rob})'\boldsymbol{P}_{t-1}^{rob}\hat{\boldsymbol{\psi}}_t^{rob}}\right\},$$

$$\hat{\boldsymbol{\varphi}}_{t+1}^{rob} = (1, (\hat{y}_t^{rob})^2, \ldots, (\hat{y}_{t+1-p}^{rob})^2, (\hat{\boldsymbol{\varphi}}_t^{rob})_t'\hat{\boldsymbol{\theta}}_t^{rob}, \ldots, (\hat{\boldsymbol{\varphi}}_{t+1-q}^{rob})'\hat{\boldsymbol{\theta}}_{t+1-q}^{rob})', \tag{4a}$$

$$\hat{\boldsymbol{\psi}}_{t+1}^{rob} = \hat{\boldsymbol{\varphi}}_{t+1}^{rob} + \textstyle\sum_{j=1}^{q}\hat{\beta}_{jt}^{rob}\hat{\boldsymbol{\psi}}_{t+1-j}^{rob},$$

$$\lambda_t = \tilde{\lambda}\cdot\lambda_{t-1} + (1-\tilde{\lambda}), \quad \lambda_0, \tilde{\lambda} \in (0,1), t \in \mathrm{N},$$

where the recursive estimates are collected in the vector $\hat{\boldsymbol{\theta}}_t^{rob}$. To complete this algorithm, one defines the outlier-corrected series $\{\hat{y}_t^{rob}\}$ as follows:

$$(\hat{y}_t^{rob})^2 = \begin{cases} (\hat{\boldsymbol{\varphi}}_t^{rob})'\hat{\boldsymbol{\theta}}_{t-1}^{rob} + \mathrm{sign}\left(y_t^2 - (\hat{\boldsymbol{\varphi}}_t^{rob})'\hat{\boldsymbol{\theta}}_{t-1}^{rob}\right)(u_{1-\alpha/2})^2\sqrt{\left[(\hat{\boldsymbol{\varphi}}_t^{rob})'\hat{\boldsymbol{\theta}}_{t-1}^{rob}\right]^2 + (\hat{\boldsymbol{\psi}}_t^{rob})'\boldsymbol{P}_{t-1}^{rob}\boldsymbol{\psi}_t^{rob}/\lambda_t} \\ \qquad \text{for } \left|y_t^2 - (\hat{\boldsymbol{\varphi}}_t^{rob})'\hat{\boldsymbol{\theta}}_{t-1}^{rob}\right| > (u_{1-\alpha/2})^2\sqrt{\left[(\hat{\boldsymbol{\varphi}}_t^{rob})'\hat{\boldsymbol{\theta}}_{t-1}^{rob}\right]^2 + (\hat{\boldsymbol{\psi}}_t^{rob})'\boldsymbol{P}_{t-1}^{rob}\boldsymbol{\psi}_t^{rob}/\lambda_t}, \\ y_t^2 \qquad \text{otherwise.} \end{cases} \tag{4b}$$

Note that $\hat{y}_t^{rob} = \mathrm{sign}(y_t)\cdot\sqrt{(\hat{y}_t^{rob})^2}$ and that $u_{1-\alpha/2}$ denotes the corresponding quantile of the standard normal distribution, where one usually puts $\alpha = 0.05$. The *forgetting factor* $\{\lambda_t\}_{t\in\mathrm{N}}$ is a deterministic sequence of positive real numbers less or equal to one. It represents the observation weight over time. One commonly puts $\lambda_0 = 0.95$ and $\tilde{\lambda} = 0.99$. The initialization of the algorithm is thoroughly discussed in Hendrych and Cipra (2018). Finally, one can introduce a simple projection, which completes the algorithm (4) and ensures that it will not degenerate:

$$\left[\hat{\boldsymbol{\theta}}_t\right]_{\boldsymbol{D}} = \begin{cases} \hat{\boldsymbol{\theta}}_t & \text{if } \hat{\boldsymbol{\theta}}_t \in \boldsymbol{D}, \\ \hat{\boldsymbol{\theta}}_{t-1} & \text{if } \hat{\boldsymbol{\theta}}_t \notin \boldsymbol{D}, \end{cases} \tag{5}$$

where $\boldsymbol{D} = \left\{ \theta \in \mathrm{R}^{p+q+1} \middle| \, \tilde{\delta}_1 \le \theta_1 \le \tilde{\Delta}_1; \theta_i \ge 0, i = 2, \ldots, p+q+1; \sum_{j=2}^{p+q+1} \theta_j \le 1 - \tilde{\delta}_2 \right\}$ and one usually puts $0 < \tilde{\delta}_1 \le \tilde{\Delta}_1 < \infty, 0 < \tilde{\delta}_2 < 1$, e.g. $\tilde{\delta}_1 = \tilde{\delta}_2 = 10^{-9}$ and $\tilde{\Delta}_1 = 10^2$.

The suggested algorithm (4) is inspired by the robustified version of Kalman filter derived in Cipra and Hanzák (2011). Point out that the assumption of normality can be replaced by other distributions. The performance of this algorithm was (fairly) compared by means of the various Monte Carlo experiments, which confirm its adequacy.

Parallelly, one can construct the robust recursive prediction of volatility, namely the one-step ahead prediction has the form (compare with (3)):

$$(\hat{\sigma}_{t+1|t}^{rob})^2 = \hat{\omega}_t^{rob} + \sum_{i=1}^{p} \hat{\alpha}_{it}^{rob} (\hat{y}_{t+1-i}^{rob})^2 + \sum_{j=1}^{q} \hat{\beta}_{jt}^{rob} (\hat{\sigma}_{t+1-j}^{rob})^2 \tag{6}$$

2 Real data applications

Figure 1 plots the log-returns of the daily currency rate CHF/EUR for the period from January 2000 to May 2017 which shows an apparent burst of volatility in January 2015 (the initial segment of the data is not displayed due to initialization of the recursive algorithm: the recursive estimates generally tend to be volatile here).[1] It has a clear explanation, i.e. the end of currency regulation of CHF by the Swiss National Bank since 2015*:* CHF was pegged to the Euro for around two years, with the minimum rate (or the floor) at 1.2. As of 15th January 2015, this link has been removed (as the consequence of appreciation of USD against EUR and of CHF weakening against USD).

It was natural to apply the suggested algorithm (4) to handle this time series. This estimation scheme has indeed identified the value of January 2015 as an outlier and corrected it in a proper way. Figure 2 with the recursively estimated parameters of the GARCH(1, 1) model displays that one should not ignore the presence of outlier; otherwise there occurs a jump in the estimated parameters (when one compares with non-robustified recursive estimates as in

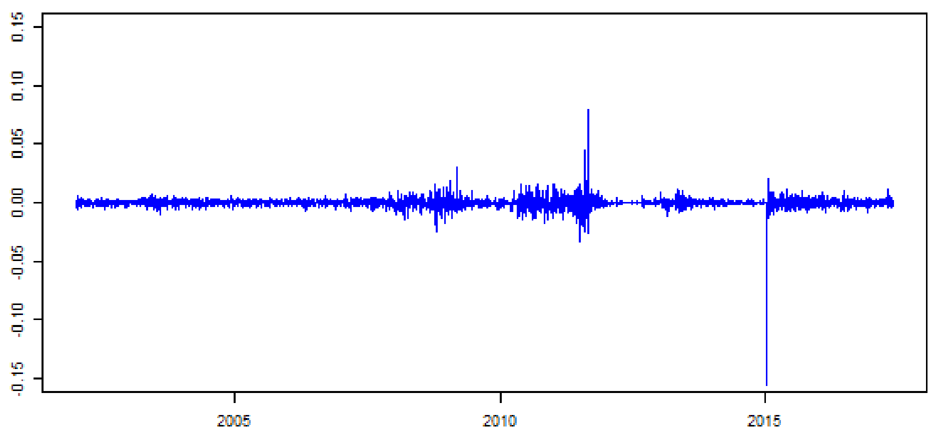
[1] www.ecb.europa.eu/stats/policy_and_exchange_rates/euro_reference_exchange_rates/html/index.en.html

Hendrych and Cipra (2018)). Moreover, another legitimate reason for the application of the algorithm (4) is that it enables the adaptation in the case of parameter changes. Similarly, the one-day ahead predictions of volatility (refer to (6)) would be out of reality without the robustification (see Figure 3).

The estimation algorithm (4) has been further applied to forty currency rates */EUR (daily log-returns) and in some of them the declared robustification has been activated (see Table 1). The results for these daily log-returns are not reported here since the figures are just of the type presented herein.

Apparently, the considered real data applications have demonstrated that the proposed robustifying modification of the recursive estimation scheme introduced in Hendrych and Cipra (2018) can (significantly) improve the quality of volatility forecasts when additive outliers are present in the financial time series.

Fig. 1: Log-returns of daily currency rate CHF/EUR (January 2002 - May 2017)



Source: Authors (calculated by statistical software **R**)

Tab. 1: Times of activation of robustification in (4) for some daily currency rates */EUR

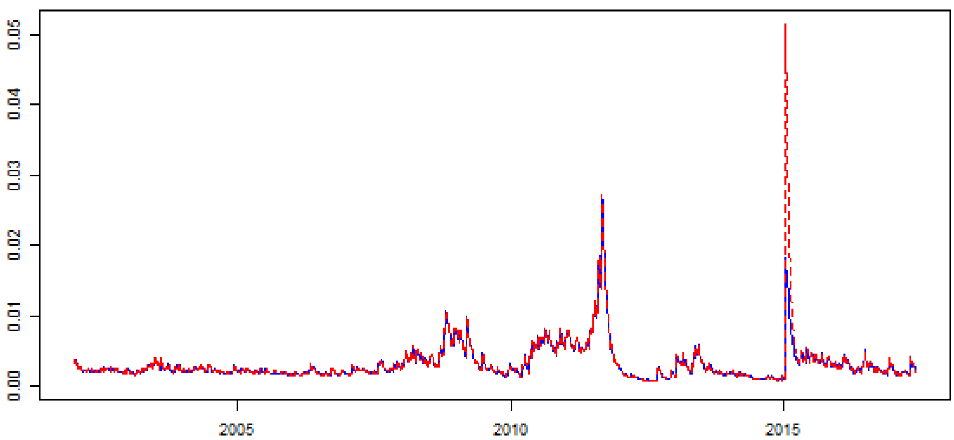
Currency	Months of identified outliers	Currency	Months of identified outliers
USD	1999−07−26	*TRY*	2006−05−12
HUF	2003−01−17	*CAD*	2000−01−04
ROL	2000−01−04	*CNY*	2006−01−23
RON	2006−05−15	*MYR*	2006−04−18
CHF	2015−01−15	*MYR*	2008−03−17
ISK	2008−11−06	*MYR*	2008−03−20
TRL	2001−02−22	*NZD*	1999−08−25

Source: Authors (calculated by statistical software **R**)

Fig. 2: Estimated parameters ω, α_1, β_1 for daily log-returns of the currency rate CHF/EUR before (*red dashed line*) and after (*blue solid line*) the robustification

Source: Authors (calculated by statistical software **R**)

Fig. 3: One-day ahead predictions of volatility for daily log-returns of the currency rate CHF/EUR before (*red dashed line*) and after (*blue solid line*) the robustification



Source: Authors (calculated by statistical software **R**)

Conclusion

The robust recursive algorithm for the estimation parameters and the corresponding volatility prediction of the GARCH model suggested in this paper seems to be effective for financial data, especially for contaminated log-returns in the risk control and regulation when the prediction of volatility is the main concern. The one-stage recursive estimation procedure introduced for the GARCH process by Hendrych and Cipra (2018) was robustified in such a way that it can distinguish and correct outlaid bursts of volatility. The real data examples also demonstrate that the suggested procedure enables corresponding adaptations in the case parameter changes.

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