Pólyas´s Theorems on Random Walks. The Precise Role of Generating Functions in their Proofs

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Abstract

One hundred years have passed since the Hungarian mathematician George (György) Pólya published the fundamental work concerning to the problem of random walk (Pólya, 1921). The theorems of George Pólya on random walks have become a popular topic of the probability theory. They were inspiration for many other results in a lot of branches of science. Here we concentrate on the core of wanderer´s problem which is of enumerative character. Our interest is in a somewhat technical matter, the precise role of generating functions and power series in the proofs of the enumerative core of Pólya´s theorems. In some contribution to this topic we can meet the opinion that the classical theorem of N. H. Abel is necessary and by this way the Pólya´s theorems may be considered as its corollary. We will see that the Pólya´s theorems are corollaries of an easier result on power series which employs the strong property of non-negativity of the coefficients of the power series.

Key words: random walk, wandering in countable graph, Abel´s theorem

JEL Code: C02, C20, C40

Introduction

This year as was mentioned marks the centenary of the publication of the fundamental article (Pólya, 1921) which was the start for new research in the area of stochastic processes. Let us describe the problem in free form and later in chapter 1 precisely in mathematical terms. A man aimlessly and randomly wanders in a rectangular net of streets. We can ask the question: how likely is it that he returns to the starting point of his trip? One may model the net of streets and crossroads with a suitable unoriented graph, for instance the graph $G = (\mathbb{Z}^d, E)$, where $d = 2$.We can consider other situations with random walk in this graph with another value of the parameter d. G. Pólya found the surprising dependence on the integer parameter d: for $d = 1$ or $d = 2$ the wanderer returns to the starting point of his walk with the probability one, but in the dimensions $d \geq 3$ with a positive probability he never returns to the start. The following contribution is about the precise role of the generating functions and the power series in the proof of the Pólya´s theorems. It means that we concentrate on a somewhat technical matter, the core of wanderer´s problem which is rather of enumerative character than probabilistic one as it consists in counting walks in the graph. We have to emphasize that this contribution is based on (W. Feller´s classics, 1968). Other references are (Billingsley, 1995), (Y. Kochetkov 2018), (K. Lange 2015), (D. A. Levin and Y. Peres, 2010), (J. Novak 2014), (K. Rogers, 2017) and (W. Woess, 2000). The list could be much extended.

1 The basic notions and denotations

As was mentioned in the introduction we will consider the graph $G = (\mathbb{Z}^d, E)$. The symbol $\mathbb{Z}$ denotes as usually the set of all integers, i.e. the set $\mathbb{Z}^d$ (the set of vertices of *G*) is the *d-th* cartesian power of the set $\mathbb{Z}$, in other words the set of all *d*-tuples of all integers. Finally the set *E* (the set of edges of the graph *G*) is created by the unordered pairs $\{u, v\}$, where $u = (\alpha_1, \dots, \alpha_d), v = (\beta_1, \dots, \beta_d) \in \mathbb{Z}^d$ for which

$$\sum_{i=1}^{d} |\alpha_i - \beta_i| = 1.$$

It means that $\{u, v\} \in E$ if and only if

$$u - v \in \{(\pm 1,0, \dots ,0), (0, \pm 1,0, \dots ,0), \dots , (0, \dots ,0, \pm 1)\}.$$

Thus one can go from *u* to *v* and back by a unit step in the direction of one of the *d* coordinate axes.

The considered graph *G* has two important properties: it is 2*d*-regular (each vertex has the same 2*d* neighbors) and is vertex-transitive (for every pair of vertices *u*, *v* there is an automorphism $f: G \to G$, for which $f(u) = v$). These properties of *G* are important as we can see in what follows.

For any vertex $u \in \mathbb{Z}^d$ and a number $n \in \mathbb{N}_0$ we denote by

$$d_n = d_n(u) \in \mathbb{N}_0, \text{ resp. } l_n = l_n(u) \in \mathbb{N}_0, \qquad (1)$$

the number of all *walks* $W = (u_0, u_1, \dots, u_n)$ in *G* starting at the point (vertex) $u_0 = u$ and with *length* $n \in \mathbb{N}_0$ (certainly $\{u_{i-1}, u_i\} \in E$ for $i = 1, \dots, n$), resp. the number of those walks that revisit the starting point: $u_0 = u = u_j$ for some $j = 1, \dots, n$. Now for any two vertices *u*, *v* and for any $n \in \mathbb{N}_0$

$$d_n = d_n(u) = d_n(v), \text{ resp. } l_n = l_n(u) = l_n(v),$$

which means that the choice of the starting vertex is irrelevant. The former equality follows from the $2d$-regularity of the graph G while the latter one requires the vertex-transitivity. However, it is satisfied thanks to automorphism (just shift) $f(x) = x + v - u$. In the proofs of Pólya´s theorems we will suppose the starting point for a random walk is the origin of the set $\mathbb{Z}^d$, i.e. the vertex $o = (0,0, \dots, 0)$.

2 The generating functions, classical Abel´s theorem and the power series with non-negative coefficients

The concepts of the generating functions and power series are old, classical but all the time interesting and fruitful. Even if the notion of generating function belongs originally to algebra there is a wide range of its use in other branches as for instance in special forms such as probability generating function or moment generating function (P. J. Dhrymes, 1985). The goal of this contribution as was mentioned is to explain the use of the theory of power series in the proofs of Pólya´s theorems on random walk. That is why we remind two results that are important to carry out our purpose.

The articles (Novak 2014) or Rogers (2017) invoke the classical Abel´s theorem. This theorem dates to 1826:

If a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with complex coefficients a_n converges for $|z| < 1$ and if the series $\sum_{n=0}^{\infty} a_n$ converges to a sum $s \in \mathbb{C}$ (the domain of all complex numbers) then $\lim_{x \to 1-} f(x) = s$, where the limit is taken along the real line from the left to 1.

Of course the general power series may have another radius of convergence, say $0 < R < \infty$ and the center of the circle of its convergence need not be the origin, but simple linear transformation enables us to consider the center at the origin and the radius equal to 1.

Let us introduce another claim about the generating functions (power series):

Proposition GFNC (on generating function with non-negative coefficients).

If a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in \mathbb{R} \qquad (2)$$

has non-negative coefficients and converges for any $x \in [0; 1)$, *then the following equality holds*

$$\lim_{x \to 1-} f(x) = \sum_{n=0}^{\infty} a_n \qquad (3)$$

no matter whether the limit and the sum are finite or infinite.

We can observe again the strong property of the non-negativity (of the coefficients) which implies the monotonicity (of the partial sums). These properties guarantee the existence both limit and infinite sum. Particularly, for arbitrary $N \in \mathbb{N}$ and $x \in [0; 1)$ we have

$$\sum_{n=0}^{N} a_n = \lim_{x \to 1-} \sum_{n=0}^{N} a_n x^n \le \lim_{x \to 1-} \sum_{n=0}^{\infty} a_n x^n \le \sum_{n=0}^{\infty} a_n$$

and we send $N \to \infty$ on the left hand side to obtain (3).

Even if this Proposition is not the classical Abel´s theorem it can be interpreted as its special case for power series with non-negative coefficients.

3 The Pólya´s theorems on random walks and their proofs

In this chapter we give the precise formulation and proofs of well-known Pólya´s theorems on random walks from which the role of power series (generating functions) in this context will be seen. For brevity we restrict only on the cases $d = 2$ and $d = 3$.

3.1 Wandering in 2 dimensions

Theorem 1. *Take the origin as the initial vertex in* $G = (\mathbb{Z}^2, E)$. *Then*

$$\lim_{n \to \infty} \frac{l_n(o)}{d_n(o)} = \lim_{n \to \infty} \frac{l_n}{d_n} = \lim_{n \to \infty} \frac{l_n}{4^n} = 1. \qquad (4)$$

Metaphorically speaking, a random walk in $G = (\mathbb{Z}^2, E)$ *returns to the starting point with probability 1.*

Proof. The symbols in (4) have the meaning from (1). Let $W = (u_0, u_1, \dots, u_n)$ be a walk in *G* of the length $n \in \mathbb{N}_0$. Let b_n be the number of all walks *W* with $u_0 = u_n = o$ and c_n be the number of those walks with $u_0 = u_n = o$ for which $u_j \ne o$ for any $0 < j < n$. For completness we set $c_0 = 0$. These numbers are not dependent on the starting point because of the vertex transitivity of the graph *G*. The following inequalities are clear: $l_n \le d_n$, $c_n \le$

$b_n \leq d_n$ and $d_n = 4^n$ for any $n \in \mathbb{N}_0$. If we compartmentalize the walks counted by l_n by their first return to o at the step j and using the fact that $l_n \leq d_n = 4^n$ we get for any $n \in \mathbb{N}_0$ the relations

$$l_n = \sum_{j=0}^{n} c_j d_{n-j}, \text{ and so } \frac{l_n}{4^n} = \sum_{j=0}^{n} \frac{c_j}{4^j} \leq 1.$$

Therefore it suffices to prove the summation

$$\sum_{j=0}^{\infty} \frac{c_j}{4^j} = 1. \qquad (5)$$

Now we pay attention to the generating functions

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{4^n} x^n \text{ and } C(x) = \sum_{n=0}^{\infty} \frac{c_n}{4^n} x^n. \qquad (6)$$

It holds that

$$B(x) = \frac{1}{1-C(x)} = \sum_{k=0}^{\infty} \big(C(x)\big)^k. \qquad (7)$$

It can be seen formally as a relation between formal power series by a splitting a walk counted by b_n in its $k+1$ returns to o into k segments of lengths $j_1, j_2, \dots, j_k, j_1 + j_2 + \cdots + j_k = n$ counted by $c_{j_1}, c_{j_2}, \dots, c_{j_k}$. However the both series in (6) have the radius of convergence at least 1 and hence the generating functions (6) are the real functions defined certainly for $x \in [0; 1)$.

To show that (5) holds it suffices to prove with respect to (7) that

$$\lim_{x \to 1-} B(x) = +\infty. \qquad (8)$$

Indeed, then we will have $\lim_{x \to 1-} C(x) = 1$ and by the Proposition GFNC, see (3), that

$$\sum_{j=0}^{\infty} \frac{c_j}{4^j} =: C(1) = \lim_{x \to 1-} C(x) = 1.$$

To prove (8) we use again the Proposition GFNC and that is why we prove that $B(1) := \sum_{j=0}^{\infty} \frac{b_j}{4^j} = +\infty$. We do it by computing b_n. It is obvious that $b_{2n+1} = 0$. For even lengths,

$$b_{2n} = \sum_{j=0}^{n} \frac{(2n)!}{j!j!(n-j)!(n-j)!} = \binom{2n}{n} \sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}^2.$$

The first equality follows by considering all positions of *j* steps of *W* to the right, which force the same number *j* of steps to the left and the same number *n-j* for steps up and down. The possibilities are counted by the multinomial coefficient $\binom{2n}{j, j, n-j, n-j}$. The last equality follows from the well-known binomial identity $\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$. The Stirling´s formula yields the asymptotics $\binom{2n}{n} \sim c \frac{4^n}{\sqrt{n}}$ for $n \to \infty$ and a constant *c*. Hence we have $\frac{b_{2n}}{4^{2n}} \sim c^2 \frac{1}{n}$ which implies

$$\lim_{x \to 1-} B(x) = B(1) = \sum_{n=0}^{\infty} \frac{b_n}{4^n} = \sum_{n=0}^{\infty} \binom{2n}{n}^2 4^{-2n} = +\infty$$

for the harmonic series is divergent as is well-known.

3.2 Wandering in 3 dimensions

Theorem 2. *We start wandering at the origin* $o = (0,0,0)$ *in* $G = (\mathbb{Z}^3, E)$. *Then*

$$\lim_{n \to \infty} \frac{l_n(o)}{d_n(o)} = \lim_{n \to \infty} \frac{l_n}{d_n} = \lim_{n \to \infty} \frac{l_n}{6^n} < 1. \qquad (9)$$

Metaphorically speaking, a random walk in $G = (\mathbb{Z}^3, E)$ *returns to the starting point with probability less than* 1 *and it disappears in infinity without return with a positive probability.*

Proof. The symbols in this proof are defined by the same way as in the proof of Theorem 1 with the only difference that everywhere 4 is replaced by 6. For instance the form of generating functions in (6) is now $B(x) = \sum_{n=0}^{\infty} \frac{b_n}{6^n} x^n$ and $C(x) = \sum_{n=0}^{\infty} \frac{c_n}{6^n} x^n$. The essential difference now is that $B(1) := \sum_{j=0}^{\infty} \frac{b_j}{6^j} < +\infty$, i.e. the series converges. The reason to show the convergence of this series is that the relation (7) holds (whatever we set for *d*), $B(1) = \lim_{x \to 1-} B(x), C(1) = \lim_{x \to 1-} C(x)$ again by the Proposition GFNC and as soon as we obtain $B(1) = \lim_{x \to 1-} B(x) < +\infty$, we will have

$$\lim_{x \to 1-} C(x) = C(1) = \sum_{j=0}^{\infty} \frac{c_j}{6^j} = \lim_{n \to \infty} \frac{l_n}{6^n} < 1.$$

So come on to prove that the series $\sum_{n=0}^{\infty} \frac{b_n}{6^n}$ converges. We have again for odd *n* that $b_n = 0$. We find an upper bound for the fraction $\frac{b_{2n}}{6^{2n}}$. The following relations are right

$$\frac{b_{2n}}{6^n} = \frac{1}{6^{2n}} \sum_{\substack{j+k\le n \\ j,k\in\mathbb{N}_0}} \frac{(2n)!}{j!j!k!k!(n-j-k)!(n-j-k)!} =$$

$$= \binom{2n}{n} 4^{-n} \sum_{\substack{j+k\le n \\ j,k\in\mathbb{N}_0}} \left(3^{-n}\binom{n}{j,k,n-j-k}\right)^2 \le$$

$$\le \binom{2n}{n} 4^{-n} \max_{\substack{x,y,z\in\mathbb{N}_0 \\ x+y+z=n}} 3^{-n}\binom{n}{x,y,z} =$$

$$= \binom{2n}{n} 12^{-n} \binom{n}{x_0,y_0,z_0}, \qquad (10)$$

where $(x_0, y_0, z_0) = \begin{cases} (m,m,m), \ n = 3m \\ (m+1,m,m), \ n = 3m+1 \\ (m+1,m+1,m), \ n = 3m+2 \end{cases}, m \in \mathbb{N}_0.$

On the first line in (10) we counted as in the proof of Theorem 1, j is the number of steps in the walk to the right, k the number of steps up, and $n-j-k$ the number of steps back. The second line is an algebraic rearrangement. On the third line we used the fact that if $\alpha_1, \alpha_2, \dots, \alpha_p$ are non-negative real numbers satisfying $\sum_{i=1}^{p} \alpha_i = 1$, then $\sum_{i=1}^{p} \alpha_i^2 \le \max_{1\le i\le p} \alpha_i$. We intend to estimate the third term and we set $\alpha_i = 3^{-n}\binom{n}{j,k,n-k}$ and $3^n = (1+1+1)^n = \sum_{\substack{j+k\le n \\ j,k\in\mathbb{N}_0}} \binom{n}{j,k,n-j-k}$. On the fourth line we found the maximum value of the trinomial coefficients using the inequality $p!\,q! > (p-1)!\,(q+1)!$ for $p \ge q+2$.

By the Stirling´s formula for the factorial we have estimates with constants *K*, *L*

$$\binom{2n}{n} < K\frac{4^n}{\sqrt{n}}\ , \ \binom{n}{x_0,y_0,z_0} < L\frac{3^n}{n}.$$

Using these estimates on (10) we obtain

$$\frac{b_{2n}}{6^n} < K\frac{1}{\sqrt{n}} L\frac{1}{n} = Cn^{-\frac{3}{2}}.$$

Finally we come to the convergence of the series $\sum_{n=0}^{\infty} \frac{b_n}{6^n}$, since

$$B(1) = \sum_{n=0}^{\infty} \frac{b_n}{6^n} = \sum_{n=0}^{\infty} \frac{b_{2n}}{6^{2n}} < C\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < +\infty \ .$$

Conclusion

The Pólya´s theorems celebrate the centenary. They are considered as the origin of very interesting area of mathematics which have important applications in other sciences (Alemany, 1997, Dias, 2020). During this time they inspired many mathematicians to further development in this field.

Nowadays there are several ways how to prove these theorems. Here is presented a pure combinatorial enumeration accompanied by generating functions, Stirling´s formula and Riemann zeta function. In this contribution the precise role of generating function is shown. The main goal was to emphasize that only weak form of Abel´s theorem (Proposition GFNC) is sufficient in the proofs. We can see again the great power of the non-negativity which is essential for the existence of the limits and sums of the given sequences and series.

The Pólya´s theorems discovered another interesting dependence of certain quality on the dimension similarly as it is in the case of the number domains (real, complex numbers and quaternions) or the expression of the roots of polynomial in radicals (Abel-Ruffini theorem) and others.

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