ON THE EXISTENCE OF CORRIDORS OF STABILITY IN A LIQUIDITY-GROWTH MODEL

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Abstract

In the paper, a two-dimensional nonlinear dynamic model describing the development of liquidity and profit of a firm around its equilibrium is studied. There is constructed a smooth control function of the model enabling the birth of double cycles around the model´s equilibrium. The inner cycle is unstable, the outer one is stable. The cycles determine two corridors of stability. The first corridor of stability lies between the equilibrium of the model and the inner cycle, the second one lies between the inner cycle and the outer cycle. The gained results give a supplement to an open problem set up by Semmler and Sieveking in their paper (Semmler and Sieveking (1993)) with respect to how the solutions of the model behave in the space between its equilibrium and the cycle which is partly pictured in Figure 2. The achieved results are illustrated by a numerical simulation.

Key words: dynamic model, equilibrium, stability, corridor stability, cycle

JEL Code: E32, E44

Introduction

In the paper, a two-dimensional nonlinear dynamic model describing the development of liquidity and profit of a firm around its equilibrium is studied. These notions are important not only for firms indicating their economic conditions, but also for banks giving them an important information on the credit assessment of firms. There is constructed a control function of the model enabling the birth of double cycles around the model´s equilibrium. It is shown that the cycles determine two corridors of stability. The first corridor of stability lies between the equilibrium of the model and the inner cycle, the second one lies between the inner cycle and the outer cycle. The concept of corridor stability was originally proposed by Leijonhufvud (Leijonhufvud (1973)) to describe the reaction of a market economy to a harmful income shock. He probably had in his mind a small domain around an important point, for example around the equilibrium of a model, describing some considered economic processes, with the property that solutions started in this domain will approach in time the equilibrium. As Leijonhufvud described this concept rather vaguely, and because economists thought that there was a limited ability of any system to withstand larger deviation of the initial values from the equilibrium, this concept attracted rather small attention. Nowadays, this concept comes back to the attention of economists (see for example Cardim de Carvalho (2016)). Economists, utilizing modern mathematical technics, have possibilities not only to study nonlinear dynamic models enabling the birth of cycles, but also at the same time a possible arise of corridors of stability. The remaining of this paper is arranged as follows.

Section 1 talks about the motivation to study the topic mentioned in introduction. In Section 2 a model investigated in this paper is introduced and performed its complete analysis. Section 3 presents a numerical simulation of a result achieved in this paper. Chapter 4 is devoted to final notes concerning reached results and indicates their possible deepening and expansion.

1 Motivation

In this section we recall some results which were gained by Semmler and Sieveking in their paper Semmler and Sieveking (1993) which motivated us to deal with this topic. They can be useful also at comparing them with ours results reached in the present paper.

Semmler and Sieveking constructed on the basis of Lotka-Volterra predator-prey model a nonlinear dynamic model describing the development of liquidity λ and profit r of a firm of the form

$$\begin{aligned} \dot{\lambda} &= (\alpha - \beta r - \varepsilon_1 \lambda - h(\lambda, r))\lambda \\ \dot{r} &= (-\gamma + \delta\lambda - \varepsilon_2 r)r \end{aligned} \tag{1}$$

with parameters $\alpha, \beta, \gamma, \delta, \varepsilon_1, \varepsilon_2$ and a control function $h(\lambda, r)$. Describing the basic properties of the control function $h(\lambda, r)$ by only verbal specification they managed to find out that model (1) together with other specific conditions on the control function enables the existence of cycles. But the authors of the paper were not able to illustrate their theoretical results by suitable numerical simulations without a smooth explicit form of the control function $h(\lambda, r)$. In their numerical simulations they worked only with a non-smooth control function

$$h(\lambda, r) = \nu[max(\mu - \lambda, 0)\,.\,max(\varphi - r, 0)]^{\frac{1}{2}}, \tag{2}$$

which is non smooth and therefore could not be used as a control function in their model. They used at their numerical simulations the values of parameter $\alpha = 0.1, \beta = 0.6, \gamma = 0.07, \delta =$

0.7, $\varepsilon_1 = 0.045, \varepsilon_2 = 0.078$which give considering the cash ratio liquidity λ the economically relevant equilibrium values $\lambda^* = 0.12, r^* = 0.15$. They claim that for smaller values of the coefficient ν from (2) the trajectories of model (1) converge to the equilibrium $\lambda^* = 0.12, r^* = 0.15$ for any initial condition. They depicted this statement by a numerical simulation for $\nu = 0.2$ in Fig.1. Further, they claim that for bigger values of coefficient ν, for example $\nu = 0.6$, the trajectories of model (1) still converge to the equilibrium $\lambda^* = 0.12, r^* = 0.15$ for smaller initial conditions, but for larger initial condition, i.e., for further departure of λ and r from the equilibrium values, however, model (1) becomes unstable. They suppose that its trajectories finally approach some limit cycle. But they were not able to illustrate this cycle with a suitable numerical simulation. On the other hand, for initial conditions, farthest away from the equilibrium the corresponding trajectories approach a half-limit cycle from the outside. This situation is illustrated in their paper for $\nu = 0.6$ by Fig.2. The behaviour of trajectories in the space between the equilibrium λ^*, r^* and this half-limit cycle was left open by the authors.

Fig. 1: $\nu = 0.2$

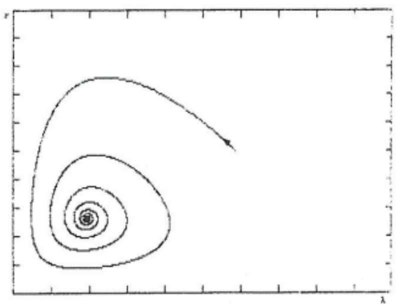
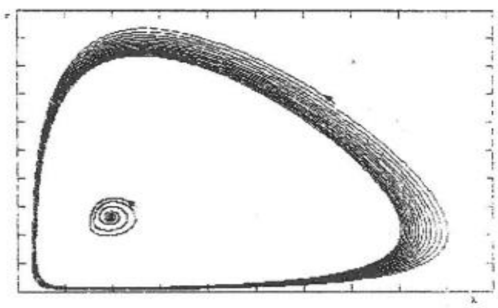


Fig. 2: $\nu = 0.6$



Source: Semmler, W. & Sieveking, M., 1993, p. 197, p. 199.

Our aim in the present paper is to find out how the solutions of a model of the type (1) may behave in the space between the equilibrium $\lambda^*, r^*$and this half-limit cycle.

2 Model

In this section we introduce and study model

$$\dot{\lambda} = \big(\alpha - \beta r - \varepsilon_1 \lambda - h(\lambda)\big)\lambda$$
$$\dot{r} = (-\gamma + \delta\lambda - \varepsilon_2 r)r, \tag{3}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon_1, \varepsilon_2$ are positive parameters and a control function $h(\lambda)$ has the form

$$h(\lambda) = k\left(\lambda - \frac{\gamma}{\delta}\right) - q(\lambda - \lambda_s)^3 - q_1(\lambda - \lambda_s)^5, \qquad (4)$$

where k, q and q_1 are parameters, and

$$\lambda_s = \frac{\beta\gamma\delta + k\gamma\epsilon_2 + \alpha\delta\epsilon_2}{\delta(\beta\delta + (k+\epsilon 1)\epsilon_2)},\ r_s = \frac{\alpha\delta - \gamma\epsilon_1}{\beta\delta + (k+\epsilon_1)\epsilon_2} \qquad (5)$$

are solutions of the equations $\alpha - \beta r - \varepsilon_1\lambda - k\left(\lambda - \frac{\gamma}{\delta}\right) = 0, \quad -\gamma + \delta\lambda - \varepsilon_2 r = 0,$ which means considering the structure of the control function (4) that λ_s and r_s are at the same time the equilibrium values of model (3). The control function $h(\lambda)$ depending only on λ is a smooth function and therefore can be used as a control function for the model (3) when studying local bifurcation.

We shall study the qualitative properties of model (3) utilizing the Bautin bifurcation theory. To apply the Bautin bifurcation theory it is required to perform the following steps (see Kuznetsov (2004)):

1. to find conditions on the parameters of model (3) which guarantee that the Jacobian matrix of model (3) has a pair of purely imaginary eigenvalues.

It can be shown using the standard procedure (see for example Wiggins (1990) or Kuznetsov (2004)) that the Jacobian matrix of model (3) is traceless if and only if

$$\alpha = \alpha_0 = -\frac{\gamma}{\delta}\frac{k(\beta\delta + \varepsilon_1\varepsilon_2) + k^2\varepsilon_2 + \delta\varepsilon_1(\beta - \varepsilon_2)}{(k + \delta + \varepsilon_1)\varepsilon_2}.$$

In the whole paper we assume that

$$k < 0, k + \varepsilon_1 < 0, k + \delta + \varepsilon_1 > 0. \qquad (6)$$

Then for small enough $\varepsilon_1, \varepsilon_2$ we have $\alpha_0 > 0$ and the Jacobian matrix of model (3) has the pair of purely imaginary eigenvalues

$$\pm i\omega_0, \quad \omega_0 = \frac{\gamma\sqrt{-k - \varepsilon_1}\sqrt{\beta\delta + (k + \varepsilon_1)\varepsilon_2}}{(k + \delta + \varepsilon_1)\sqrt{\varepsilon_2}}.$$

2. to translate the equilibrium $E = (\lambda_s, r_s)$ *into the origin* $E_0 = (0,0)$ *and the bifurcation value* α_0 *into zero by shifting*

$$x_1 = \lambda - \lambda_s, x_2 = r - r_s, \mu = \alpha - \alpha_0.$$

We gain the system

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, \quad (7)$$

where

$$\dot{x}_1 = -\frac{1}{(k+\delta+\epsilon_1)(\beta\delta+(k+\epsilon_1)\epsilon_2)}\left(kx_1 - qx_1^3 - q_1x_1^5 + x_2\beta + x_1\epsilon_1\right)$$
$$\big(\beta\gamma\delta + x_1\beta\delta(\delta+\epsilon_1) + k^2x_1\epsilon_2 + \gamma\epsilon_1\epsilon_2 + kx_1(\beta\delta+(\delta+2\epsilon_1)\epsilon_2) + k\epsilon 2(\gamma+\mu)$$
$$+ (\delta+\epsilon_1)\epsilon_2(x_1\epsilon_1+\mu)\big),$$

$$\dot{x}_2 = \frac{(x_1\delta - x_2\epsilon_2)\left(-\gamma\epsilon_1 + x_2(\beta\delta+(k+\epsilon_1)\epsilon 2) - \frac{\gamma\left(\delta\epsilon_1(\beta-\epsilon_2)+k^2\epsilon_2+k(\beta\delta+\epsilon_1\epsilon_2)\right)}{(k+\delta+\epsilon_1)\epsilon_2} + \delta\mu\right)}{\beta\delta+(k+\epsilon_1)\epsilon_2}.$$

3. to transform system (7) by the substitution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where the matrix M consists of the eigenvectors of the Jacobian matrix of system (7) at $x_1 = 0, x_2 = 0, \mu = 0$, to the form with the Jordan linear approximation matrix

$$\dot{y} = Ay + G,$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} \sigma(\mu) & 0 \\ 0 & \bar{\sigma}(\mu) \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad G_1 = \sum_{2\le k+l\le 5}\frac{1}{k!\,l!}g_{kl}(\mu)x_1^k x_2^l + O(|x|^6),$$

$$\sigma(\mu) = \eta(\mu) + i\omega(\mu), \eta(0) = 0, \omega(0) = \omega_0, \bar{\sigma}(\mu) = \eta(\mu)\text{-i}\omega(\mu), G_2 = \bar{G}_1,$$

where

$$\eta(\mu) = \frac{-k^2\epsilon_2^{3/2}\mu - 2k(\delta+\epsilon_1)\epsilon_2^{3/2}\mu - (\delta+\epsilon_1)^2\epsilon_2^{3/2}\mu}{2(k+\delta+\epsilon_1)\sqrt{\epsilon_2}(\beta\delta+(k+\epsilon_1)\epsilon_2)} \quad (8)$$

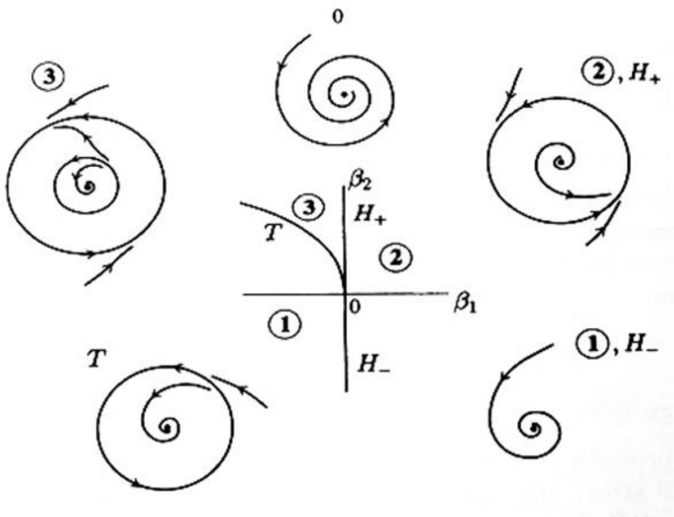
$$\omega(\mu) = (-\sqrt{}(4\gamma^2(k+\epsilon_1)(\beta\delta+(k+\epsilon_1)\epsilon_2)^3 + 4\gamma(k-\delta+\epsilon_1)(k+\delta+\epsilon_1)\epsilon_2$$
$$(\beta\delta+(k+\epsilon_1)\epsilon_2)^2\mu + (k+\delta+\epsilon_1)^2\epsilon_2^2(-4\beta\delta^2+(k-\delta+\epsilon_1)^2\epsilon_2)\mu^2))/$$
$$2(k+\delta+\epsilon_1)\sqrt{\epsilon_2}(\beta\delta+(k+\epsilon_1)\epsilon_2),$$

and the symbol $(\bar{\text{O}})$ means the complex conjugate expression to O. As the expressions G_1, G_2 are complex conjugate forms, we will further deal only with the form G_1. In our case it is not necessary to derive the Bautin normal form

$$\dot{u} = (\beta_1 + i)u + \beta_2 u|u|^2 + su|u|^4 + O(|u|^6),$$

which gives by its solution at $s = -1$ the bifurcation diagram shown in Fig. 3.

Fig. 3: Bifurcation diagram of the Bautin normal form



Source: Kuznetsov, Yu. A., 2004, p. 314.

On this bifurcation diagram there are pictured all possible qualitative behaviours of system (7)´s solutions around its equilibrium with respect to the values of parameters β_1 and β_2. Taking into account our aim, we see that only the behaviour of solutions corresponding to the pairs (β_1, β_2) lying in the space denoted by number 3 which is determined by the arc T of a parabola and the positive part of the parameter β_2. Hence, for our purpose it will be sufficient to derive only Lyapunov coefficients l_1, l_2 from the relation (8.20) in Kuznetsov (2004) and to determine the Bautin point B consisting of two bifurcation values. The first bifurcation value is the value α_0 which was shifted into the value $\mu = 0$. The second bifurcation value should be such that both bifurcation values creating the Bautin point B guarantee that $l_1(B) = 0$ and $l_2(B) \neq 0$. A suitable candidate for this role is parameter q. Lyapunov coefficient $l_1(\mu, q)$ is given by the relation

$$l_1(\mu, q) = \frac{Re\ \ c_1(\mu,q)}{\omega(\mu,q)} - \eta(\mu, q)\frac{Im\ \ c_1(\mu,q)}{\omega^2(\mu,q)}, \tag{9}$$

where

$$Re\ \ c_1(\mu, q) = \frac{3q}{2}\frac{\beta\gamma\varepsilon_2\sqrt{\beta\delta+(k+\varepsilon_1)\varepsilon_2}}{\delta(k+\varepsilon_1)(k+\delta+\varepsilon_1)}. \tag{10}$$

We get from (8) that $\eta(\mu = 0) = 0$ and from (10) that $Re\ \ c_1(\mu, q) = 0$ at $q = 0$. Hence we can use the value $q = 0$ for the second bifurcation value and the point $B = (\mu = 0, q = 0)$ as the Bautin bifurcation point because $l_1(B) = 0$.

Let us calculate the second Lyapunov coefficient $l_2(B)$ which is given in Kuznetsov (2004) by the formula (8.23) consisting of the coefficients $g_{kl}(\mu)$ of the Taylor expansion of the function G_1. As all the coefficients $g_{kl}(\mu)$ of the third and the fourth degree are zeros at $\mu = 0$ and $q = 0$, we get from the formula (8.23) the expression

$$12\ l_2(B) = \frac{1}{\omega_0}Re\ \ g_{32} + \frac{1}{\omega_0^4}\{Im[g_{11}\bar{g}_{02}(\bar{g}_{20}^2) - 3\bar{g}_{20}g_{11} - 4g_{11}^2] + Im(g_{20}g_{11})\,[3\,Re\ \ (g_{20}g_{11}) - 2|g_{02}|^2]\},$$

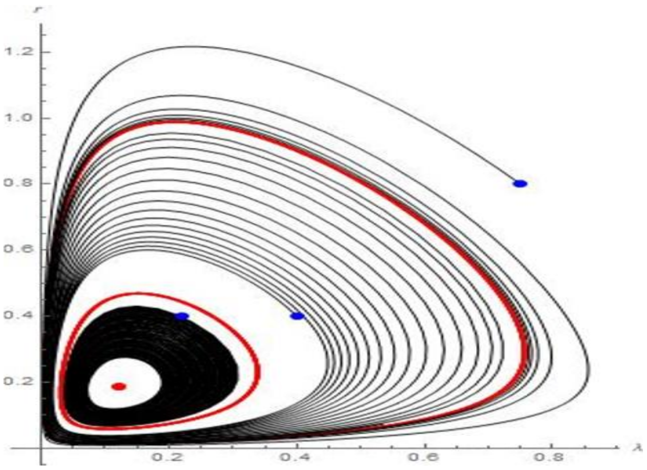
where all the g_{kl} are evaluated at the point $B = (\mu = 0, q = 0)$. We get

$$l_2(B) = \Big\{\frac{-1}{12}\epsilon_2^{5/2}(-60q_1(\beta\delta - \sqrt{-k-\epsilon_1}\epsilon_2)(\beta\delta + 2(k+\epsilon_1)\epsilon_2)^2((k+\epsilon_1)\sqrt{\epsilon_2} - \sqrt{-k-\epsilon_1}$$
$$\sqrt{\beta\delta + (k+\epsilon_1)\epsilon_2}\,\frac{1}{2\gamma^4\delta^2}\sqrt{-k-\epsilon_1}(k+\delta+\epsilon_1)^4\sqrt{\beta\delta+(k+\epsilon_1)\epsilon_2}(-2(k+\epsilon_1)\epsilon_2 - \delta(\beta+\epsilon_2))$$
$$(6\beta\delta^5\epsilon_2 - 8\delta^3(k+\epsilon_1)^2(3\beta - 4\epsilon_2)\epsilon_2 - 24\delta^2(k+\epsilon_1)^3\epsilon_2^2 + 12\delta^3(k+\epsilon_1)\epsilon_2(\beta\delta + (k+\epsilon_1)\epsilon_2) +$$
$$2\delta^4(k+\epsilon_1)(\beta^2 + 14\beta\epsilon_2 + 4\epsilon_2^2) + \sqrt{-k-\epsilon_1}\sqrt{\epsilon_2}\sqrt{\beta\delta+(k+\epsilon_1)\epsilon_2}(-2(k+\epsilon_1)\epsilon_2 - \delta(\beta+\epsilon_2))$$
$$\left(3\beta\delta^3\epsilon_2 - 4\delta(k+\epsilon_1)^2(3\beta-4\epsilon_2)\epsilon_2 - 12(k+\epsilon_1)^3\epsilon_2^2 + \delta^2(k+\epsilon_1)(\beta^2 + 14\beta\epsilon_2 + 4\epsilon_2^2)\right))))/$$
$$\left(8\delta^4(k+\epsilon_1)^3(\beta\delta - \sqrt{-k-\epsilon_1}\epsilon_2)^2\right))\Big\}.$$

3 Numerical simulation

We showed that at the Bautin point $B = (\mu = 0, q = 0)$ there is $l_1(B) = 0$, what is the first condition for getting a double limit cycle. The second condition which should be satisfied is $l_2(B) \neq 0$. It can be shown that at the values $\beta = 0.6, \gamma = 0.07, \delta = 0.7, \epsilon_1 = 0.045, \epsilon_2, = 0.078$, $k = -0.16, q_1 = 0.7$ there is $l_2(B) = -4.2896$, what means that the second condition is also satisfied at these values of parameters. The corresponding double cycle is pictured in Fig, 4.

Fig. 4



Source: the authors.

This picture gives an answer to the open problem of how the solutions of system (3) behave in the space between its equilibrium and the outer cycle. At the same time, we see that in this space there are two corridors of stability. The first one is situated in the space between the equilibrium of system (3) and the inner unstable cycle, the second one is situated between the inner cycle and the outer stable cycle.

4 Conclusion

In this paper there is solved the open problem set up by Semmler and Sieveking (Semmler and Sieveking (1993)) on the unknown behaviour of model (1)´s solutions in the space between its equilibrium and half-cycle stable from outside (see Fig. 2). In the present paper this open problem is cleared up and illustrated in Fig. 4. The Bautin bifurcation theory enables to expand the gained results also to other qualitative behaviours of model (3)´s solutions as is indicated on bifurcation diagram of the Bautin normal form.

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