ON THE EXISTENCE OF STABILITY CORRIDORS IN A MODEL OF CENTRAL BANKING

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Abstract

Dealing with macroeconomic processes, we are quite often interested in their qualitative behavior in time. For practice, especially a case of their behavior in a neighborhood of the corresponding equilibrium is important. It is shown that a utilization of the bifurcation theory of differential equations enables to study such issues. In the paper there is presented a case of possible development of nominal rate of interest and expected rate of inflation in an economy which is described by a two-dimensional nonlinear differential system with respect to continuous time. The conditions guaranteeing the birth of double cycles around the system´s equilibrium are found using Bautin local bifurcation theory. In the case of two cycles the inner cycle is unstable, the outer one is stable. The cycles determine two corridors of stability. The first corridor of stability lies between the stable equilibrium of the system and the inner cycle, the second one lies between the inner cycle and the outer cycle. The results are illustrated by numerical simulation.

Key words: dynamic model, equilibrium, stability, limit cycle, corridor stability

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Introduction

In recent times, the credibility of Minsky's financial instability hypothesis (Minsky, 1982, 1986) which means that a financially dominated capitalist economy is inherently unstable, is growing rapidly. It seems that recent turbulence in the world economy has proved it. For example, the Japanese economy experienced serious deflationary depression in the 1990s and the 2000s, and serious financial crisis, which began in the USA with the 2008 mortgage crisis, rapidly spread to other parts of the world such as Europe and Asia. But Minsky did not think that such inherent instability was uncontrollable by the government and the central bank. He emphasized that it is important to 'stabilize an unstable economy' by means of proper macroeconomic stabilization policies implemented by the government and the central bank. In this respect, Minsky inherits Keynes' spirit (Keynes, 1936).

As a reaction especially to the deflationary depression in the Japanese economy, Asada (2011) set up a simple Keynesian macrodynamic model of monetary policy describing the development of nominal rate of interest and expected rate of inflation. In Asada (2011), however, analytical treatment is rather sketchy, and the numerical simulation is not presented. In the paper Asada et al (2016) the dynamic model from Asada (2011) is studied rigorously both analytically and numerically.

In the present paper, Asada´s (2011) two-dimensional nonlinear dynamic model is studied again from the point of view of the existence of a stability corridor. Economists, utilizing modern mathematical technics, have possibilities not only to study nonlinear dynamic models enabling the birth of cycles, but also at the same time possible arises of stability corridors. For this purpose, we enriched Asada´s dynamic model by a control function enabling the birth of double cycles around the model´s equilibrium. It is shown that the cycles determine two corridors of stability. The first corridor of stability lies between the equilibrium of the model and the inner cycle, the second one lies between the inner cycle and the outer cycle. Issues dealing with the existence of stability corridors are studied also for example in Leijonhufvud (1973), Semmler and Sieveking (1993), Diamond (2005) and Cardim de Carvalho (2016).

This paper is arranged as follows. A motivation to study the topic of the existence of stability corridors is explained in Introduction. In Section 1 a model investigated in this paper is introduced and performed its complete analysis. Section 2 presents a numerical simulation of a result achieved in this paper. Section 3 is devoted to final notes concerning reached results and indicates their possible deepening and expansion.

1 Model

The original Asada´s model which was set up by Asada in Asada (2011) has the form

$$\dot{r} = \alpha(\pi - \overline{\pi}) + \beta(Y - \overline{Y}) \tag{1}$$

$$\dot{\pi}^e = \gamma[\theta(\overline{\pi} - \pi^e) + (1 - \theta)(\pi - \pi^e)] \tag{2}$$

$$Y = Y(r - \pi^e, G, \tau), \quad \pi = \varepsilon(Y - \overline{Y}) + \pi^e, 0 < \tau < 1, 0 \leq \theta \leq 1, \tag{3}$$

Y- real national income, $\overline{Y}$- natural output level corresponding to the natural rate of unemployment (fixed), G- real government expenditure (fixed), τ- marginal tax rate (fixed), $\pi = \frac{\dot{p}}{p}$ – the rate of inflation, π^e- expected rate of inflation, $\overline{\pi}$- target rate of inflation, r- nominal rate of interest, $r - \pi^e$-expected real rate of interest, $\alpha, \beta, \gamma, \varepsilon > 0$.

Putting relations (3) into equations (1) and (2) we receive

$$\dot{r} = \alpha\big(\varepsilon(Y(r-\pi^e, G, \tau) - \bar{Y}) + \pi^e - \bar{\pi}\big) + \beta\big(Y(r-\pi^e, G, \tau) - \bar{Y}\big) \tag{4}$$

$$\dot{\pi}^e = \gamma\big(\theta(\bar{\pi} - \pi^e) + (1-\theta)\varepsilon(Y(r-\pi^e, G, \tau) - \bar{Y})\big). \tag{5}$$

In Asada (2011), however, analytical treatment is rather sketchy, and the numerical simulation is not presented. In the paper Asada et al (2016) the dynamic model from Asada (2011) is studied rigorously both analytically and numerically.

To study the existence of stability corridors it is suitable to enrich model (4)-(5) by a control function

$$h(\pi^e) = \eta_3(\bar{\pi} - \pi^e)^3 + \eta_5(\bar{\pi} - \pi^e)^5. \tag{6}$$

The control function $h(\pi^e)$ with real parameters $\eta_3$end η_3 is smooth function and therefore can be used as a control function for model (4) - (5). Adding it to model (4)-(5) we get the model

$$\dot{r} = \alpha\big(\varepsilon(Y(r-\pi^e, G, \tau) - \bar{Y}) + \pi^e - \bar{\pi}\big) + \beta\big(Y(r-\pi^e, G, \tau) - \bar{Y}\big)$$
$$\equiv\ f_1(r, \pi^e, \alpha, \beta, \varepsilon, G, \tau) \tag{7}$$

$$\dot{\pi}^e = \gamma(\theta(\bar{\pi} - \pi^e) + (1-\theta)\varepsilon(Y(r-\pi^e, G, \tau) - \bar{Y}) + \eta_3(\bar{\pi} - \pi^e)^3 + \eta_5(\bar{\pi} - \pi^e)^5)$$
$$\equiv f_2(r, \pi^e, \gamma, \theta, \varepsilon, G, \tau). \tag{8}$$

The normal equilibrium point $E = (r^*, \pi^{e*})$ of this system is determined by the relations $\dot{r} = 0, \dot{\pi}^e = 0, Y = \overline{Y}$. From the structure of the functions $f_1(r, \pi^e; \alpha, \beta, \varepsilon, G, \tau)$ and $f_2(r, \pi^e; \gamma, \theta, \varepsilon, G, \tau)$ we receive

$$Y(r^* - \overline{\pi}, G, \tau) = \overline{Y}, \tag{9}$$

$$\pi^{e*} = \pi^* = \overline{\pi}. \tag{10}$$

Equation (9) means that the ´natural´ output level is realized at the normal equilibrium point. Equation (10) means that the expected rate of inflation is realized, and the realized rate of inflation is equal to the target rate of inflation at the normal equilibrium point.

The nominal rate of interest at the normal equilibrium point r^* is determined as follows. First, the equilibrium real rate of interest ρ^* is determined by the equation $Y(\rho^*, G, \tau) = \overline{Y}$. Solving this equation with respect to ρ^*, we have

$\rho^* = \rho^*(G,\tau);\ \frac{\partial\rho^*}{\partial G} > 0, \frac{\partial\rho^*}{\partial\tau} < 0.$ (11)

Then we have

$r^* = \rho^*(G,\tau) + \bar{\pi}.$ (12)

If r^* in (12) is negative, the economically meaningful normal equilibrium point does not exist. The condition $r^* > 0$ is equivalent to the condition

$\bar{\pi} > -\rho^*(G,\tau).$ (13)

If G is sufficiently small and/or τ is sufficiently large, the equilibrium real rate of interest ρ^* may become negative. In this case, inequality (13) may not be satisfied so that the normal equilibrium point need not exist if the target rate of inflation $\bar{\pi}$ is not sufficiently large. From now on, we assume that inequality (13) is satisfied so that the equilibrium nominal rate of interest r^* is positive. We shall study the qualitative properties of model (7) - (8) utilizing the Bautin bifurcation theory. To apply the Bautin bifurcation theory it is required to perform the following steps (see Kuznetsov (2004)):

1. to find conditions on the parameters of model (7)-(8) which guarantee that the Jacobian matrix of model (7)-(8) has a pair of purely imaginary eigenvalues.

It can be shown using the standard procedure (see for example Wiggins (1990) or Kuznetsov (2004)) that the Jacobian matrix of model (7) - 8) is traceless if and only if the value of parameter β is

$$\beta_0 = \varepsilon[\gamma(1-\theta) - \alpha] + \frac{\gamma\theta}{Y_{r^*-\pi^{e*}}}, \quad Y_{r-\pi^e} = \frac{\partial Y}{\partial(r-\pi^e)}.$$

The value β_0, called the bifurcation value, is positive for θ from the interval $0 < \theta < \theta_m$, where

$$\theta_m = \frac{(\alpha-\gamma)\varepsilon Y_{r^*-\pi^{e*}}}{\gamma - \gamma\varepsilon Y_{r^*-\pi^{e*}}} \quad .$$

For the positive bifurcation values β_0 the Jacobian matrix of model (7)-(8) has the pair of purely imaginary eigenvalues

$$\pm i\omega, \omega = \sqrt{-\gamma[\gamma\theta^2 - \epsilon(-1+\theta)(\alpha+\gamma\theta)Y'(r^*-\pi^*)]}.$$

2. to translate the equilibrium $E = (r^*, \pi^{e*})$ into the origin $E_0 = (0,0)$ and the bifurcation value β_0 into zero by shifting $x_1 = r - r^*,\ x_2 = \pi^e - \pi^{e*},\ \mu = \beta - \beta_0$. We gain the system

$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix},$ (14)

which is for enrolment rather long and not enough lucid. Hence, to be able to present system (14) in detailed form, we rewrite system (7)-(8) in the following re-parametrization

$C_1 = \alpha(\varepsilon+\beta)(Y-\bar{Y}), C_2 = \alpha, C_3 = \gamma\theta, C_4 = \gamma(1-\theta)\varepsilon.$

In this set up system (7)-(8) has the form

$$\dot{r} = C_1\big(Y-\bar{Y}) + C_2(\pi^e - \bar{\pi})\big)$$

$$\dot{\pi}^e = C_3(\bar{\pi}-\pi^e) + C_4(Y-\bar{Y}) + \eta_3(\bar{\pi}-\pi^e)^3 + \eta_5(\bar{\pi}-\pi^e)^5.$$

Introducing the production function Y in the form $Y = \frac{A}{b} + \frac{k}{e^{r-\pi^e}}$ with suitable positive constants A, b, k, system (7)-(8) takes the form

$$\dot{x}_1 = C_1\left[\frac{A}{b} - \bar{Y} + \frac{1}{b}(-A+b\bar{Y})e^{-x_1+x_2}\right] + C_2x_2 \qquad (15)$$

$$\dot{x}_2 = C_4\left[\frac{A}{b} - \bar{Y} + \frac{1}{b}(-A+b\bar{Y})e^{-x_1+x_2}\right] - C_3x_2. \qquad (16)$$

3. to transform system (15)-(16) by the substitution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where the matrix M consists of the eigenvectors of the Jacobian matrix of system (15)-(16) at $x_1 = 0, x_2 = 0, \mu = 0$, to the form with the Jordan linear approximation matrix

$$\dot{y} = By + G,$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad B = \begin{pmatrix} \sigma(\mu) & 0 \\ 0 & \bar{\sigma}(\mu) \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad G_1 = \sum_{2\le k+l\le 5} \frac{1}{k!\,l!} g_{kl}(\mu)x_1^k x_2^l + O(|x|^6),$$

$\sigma(\mu) = \xi(\mu) + i\omega(\mu), \xi(0) = 0, \omega(0) = \omega_0, \bar{\sigma}(\mu) = \xi(\mu)\text{-i}\omega(\mu), G_2 = \bar{G}_1,$

where

$$\xi(\mu) = \frac{1}{2b}(AC_1 - b\bar{Y}C_1 - bC_3 - AC_4 + b\bar{Y}C_4)$$

$$\omega(\mu) = \frac{1}{2b}\big[4\big(-AbC_1C_3 + b^2\tilde{Y}C_1C_3 - AbC_2C_4 + b^2\tilde{Y}C_2C_4\big)$$

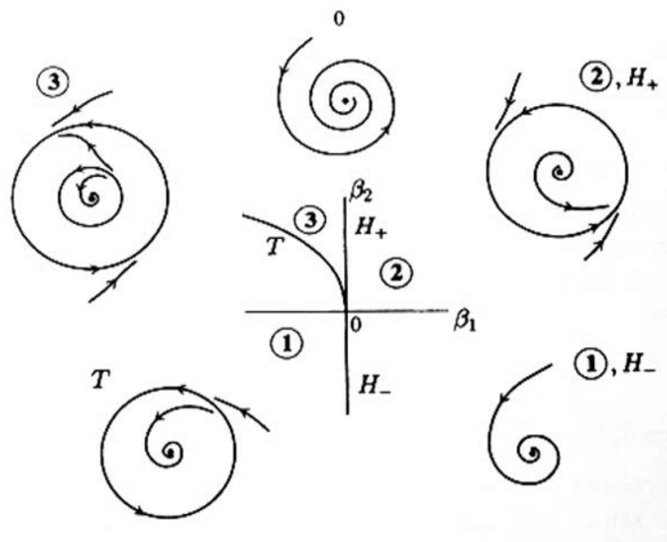
$$-(-AC_1 + b\bar{Y}C_1 + bC_3 + AC_4 - b\bar{Y}C_4)\big]^{\frac{1}{2}},$$

and the symbol $(\bar{O})$ means the complex conjugate expression to O. As the expressions G_1, G_2 are complex conjugate forms, we will further deal only with the form G_1. In our case it is not necessary to derive the Bautin normal form

$$\dot{u} = (\beta_1 + i)u + \beta_2 u|u|^2 + su|u|^4 + O(|u|^6),$$

which gives by its solution at $s = -1$ the bifurcation diagram shown in Fig. 1.

Fig. 1: Bifurcation diagram of the Bautin normal form



Source: Kuznetsov, Yu. A., 2004, p. 314.

On this bifurcation diagram there are pictured all possible qualitative behaviours of system (7)-(8)´s solutions around its equilibrium with respect to the values of parameters β_1 and β_2. Taking into account our aim, we see that only the behaviour of solutions corresponding to the pairs (β_1, β_2) lying in the space denoted by number 3 which is determined by the arc T of a parabola and the positive part of the parameter β_2.

2 Numerical simulations

In this chapter we present numerical simulations obtained by the analysis of system (7)-(8). We take the following values of parameters:

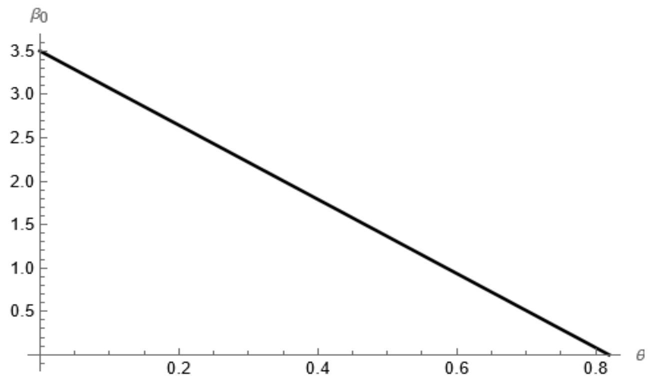
$\bar{\pi} = 0.02, \bar{Y} = 120, T_0 = 2, C_0 = 5, G = 52, c = 0.8, \tau = 0.4,$

which give the equilibrium of system (7)-(8): $E = (r^* = 0.05, \pi^{e*} = 0.02)$. Then the constant k is given by the relation

$k = \{[1 - c(1 - \tau)]\bar{Y} - (cT_0 + C_0 + G)\}e^{r^* - \pi^{e*}}.$

Further, we take the adjustment constants $\alpha = \frac{1}{4}, \varepsilon = 2, \gamma = 2$. Now we are able to obtain a critical value β_0 as a function of the credibility parameter θ. Admissible range of θ is $0 \leq \theta \leq \theta_m, \theta_m \doteq 0.82$ The result is shone in Fig. 2.

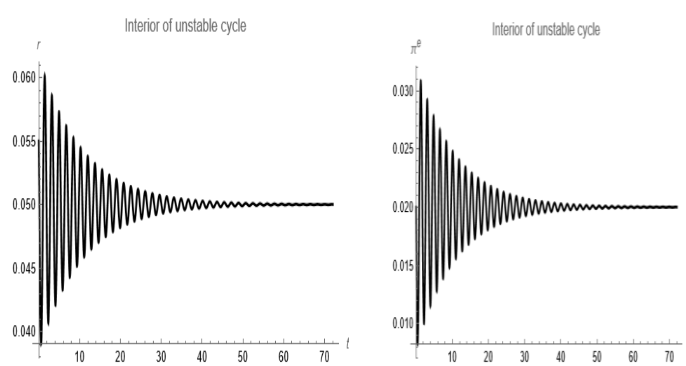
Fig. 2: Admissible range of the credibility parameter θ



Source: the authors.

We are interested in the behaviour of solutions in a neighbourhood of the equilibrium of model (7)-(8) with respect to the neighbourhood of the point in parametric space:$(\beta, \eta_3) = (\beta_0, 0) \equiv B_p$. The first Lyapunov coefficient l_1 vanishes at B_p and the second Lyapunov coefficient is non-zero for every non-zero η_5. Therefore, our system is locally equivalent with the Bautin normal form, which provide the case of coexistence of stable equilibrium, inner unstable cycle and outer stable cycle. In our case we fix the parameters as follows: $\beta = 1.05 \times \beta_0, \eta_3 = 10 \times cos\,\Phi\,, \eta_5 = 40 \times sin\,\Phi$, where $\Phi = 0.65 \times \Pi$. Dynamics is shown on Figures bellow.

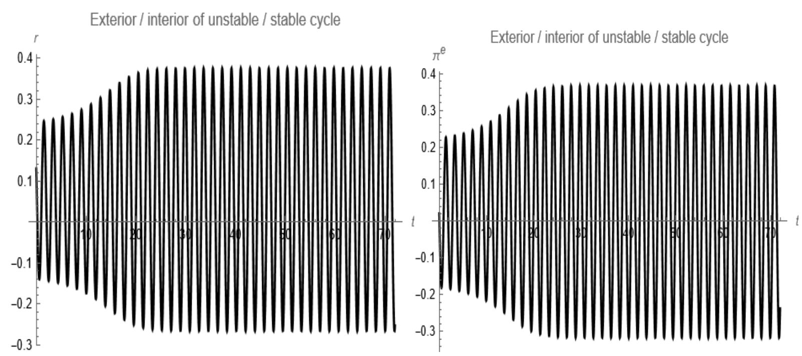
Fig. 3: Evolution of variables r and π^e starting in the interior of the unstable cycle with initial values $r(0) = 1.1 \times r^*, \pi^e(0) = \pi^{e*}$.



Source: the authors.

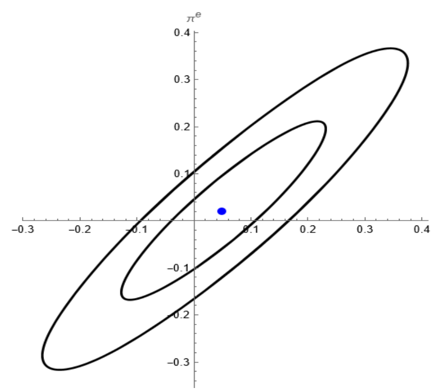
Evolution of variables r and π^e starting in between inner unstable cycle and outer stable cycle with the initial values $r(0) = 2.6 \times r^*, \pi^e(0) = \pi^{e*}$. Dynamics is shown on Figures bellow.

Fig. 4: Evolution of variables r and π^e starting in between inner unstable cycle and outer stable cycle with the initial values $r(0) = 2.6 \times r^*, \pi^e(0) = \pi^{e*}$.



Source: the authors.

Fig. 5: Stable outer with unstable inner cycles and the equilibrium as blue point



Source: the authors.

Conclusion

In the paper there is presented the case of the behaviour of solutions in a neighbourhood of model (7) - (8)´s equilibrium enabling the coexistence of two cycles, the inner one unstable and the outer one stable. As to the future analyses of the model, it could be interesting to analyse also other cases of qualitative behaviour of solutions which Bautin bifurcation enables.

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