BIRTH OF THE TERM FUNCTION

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Abstract

The first examples of functional thinking and mathematical expression of dependencies come from Babylonia in the 2nd to 1st millennium BC. The Greeks in the 6th to 5th centuries BC were aware of the difference between discrete and continuous quantities, but the idea of a general function or variable quantity did not appear anywhere. The idea of the function first appeared in the Middle Ages. In the 5th century AD, the sine function appears in India. The Arab al-Biruni thought of a general curve alone. In Europe, in 14th century, Oresme described functional dependence in a verbally expressed rule or graphically. At the turn of the 16th and 17th centuries, the Napier discovered logarithms, Descartes began to use the terms independent variable and dependent variable, Newton and Leibniz laid the foundations of infinitesimal calculus. The definition of function was first formulated in 1718 by Johann Bernoulli. In the 19th century, Lobachevsky and Dirichlet's conception of a function as an unambiguous assignment of a dependent variable to an independent variable that is considered classical today. The emerging set theory at the turn of the 19th and 20th centuries made it to view functions as a representation of one set into another.

Key words: Function, Newton, Leibniz, Dirichlet, Lobachevsky

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Introduction

The concept of function is one of the most important and widely used concepts in mathematics. Math is laced with functions, any symbol that appears in it mostly representing a function. We can liken the function to a coffee machine. What makes such a machine typical? If we put coins giving some currency value into the machine, the machine issues us some kind of coffee. If we drop coins corresponding to the higher value of the currency into the machine, the machine will respond with a better kind of coffee. The function represents such a mathematical coffee machine. You insert some input into the feature (such as a coin machine) and the function returns some output (the machine issues a type of coffee). That's the whole basic principle of

function. The function's job is to take some input (number), do something with this number, change it and then issue this new number as output.

Today, a function is a special type of dependency in which each value from one set (the domain of the function) is assigned exactly one value from the other set (the range of the function).

In studying the phylogenesis of the concept of function, you can see that the concept has been very slow and hard to get into mathematics. The wording of the definition itself, and a broader view of functional dependency, preceded centuries of causal thinking development and thorough work with specific function representatives. It is the very valuable lessons that can be found in the history of mathematics.

1 Ancient times and Middle Ages

The first evidence of functional thinking and mathematical expression of dependencies came from Babylon in the 2nd to 1st millennium BC. The tables of functions that Babylonians used in astronomy, agriculture, construction, trade and tax calculation were preserved. This was about, for example, functions $n \dashrightarrow \frac{1}{n}$, $n \dashrightarrow n^2$, $n \dashrightarrow \sqrt{n}$, $n \dashrightarrow n^3$, $n \dashrightarrow \sqrt[3]{n}$, $n \dashrightarrow n^2+n$, kde $n \in N$ (Bourbaki, 1994).

The Greeks, in the 6th to 5th centuries BC, recognized the difference between discrete and continuous quantities, developed infinitesimal reflections, studied specific changes in variable quantity (movement), but nowhere did the idea of a general function or variable quantity stand out. Pythagoreans tried to express the relation of length and diameter of the string with the pitch of the sound.

Geometrists studied the curves created by the continuous movement of a point and formulated the kinematic laws of their formation. Menaechmos and Apollonios, in the 4th to 2nd centuries BC, tried to verbally describe broader curves in geometric terms (symptoms) as any ellipse, parabola, hyperbole.

Diocles and Nikomédes discovered in the 2nd century BC in connection with solving the problem of doubling of the cube and trisection angle of the curves of the kisoid and the conchoid, Ptolemy in the 2nd century AD tabulated the chordal (Bourbaki, 1994).

The idea of the function first emerged in the Middle Ages. Indians in the 5th century AD began to use the *sinus* function. The Arabic mathematician al-Birouni (973 – 1048) led reflections on the general curve, but in this his done (Bourbaki, 1994).

In the 12th century, the Arabic mathematician Sharaf al-Din al-Tusi analysed the equation $x^3 + \mathrm{d} = bx^2$ in the form $x^2(b - x) = d$, stating that the left-hand side must at least equal the value of d for the equation to have a solution. He then determined the maximum value of this expression. It is arguable that the isolation of this expression is an early approach to the notion of a "function". A value less than d means no positive solution; a value equal to d corresponds to one solution, while a value greater than d corresponds to two solutions. Sharaf al-Din's analysis of this equation was a notable development in Islamic mathematics, but his work was not pursued any further at that time, neither in the Muslim world nor in Europe.

In Europe (Juškevič, 1970), the legacy of Greek and Arabic mathematics was exploited in the 11th to 15th centuries, Arabic and Greek mathematical works were translated into Latin, numerical techniques were deepened, the Indian-Arabic numerical system was used, symbols + and – were gradually added. Mathematics processed change and movement. Scholastics Robert Grosseteste (circa 1168 – 1253), Thomas Bradwardinus (circa 1290 – 1649), Nicole Oresme (circa 1323 – 1382) and Richard Swineshead (14th century) at the Universities of Oxford and Paris described and examined natural events and movements, beginning to consider a continuum.

Grosseteste is best known as an original thinker for his work concerning what would today be called science or the scientific method. Hence, since light could be reduced to lines and points, and thus fully explained in the realm of mathematics, mathematics was the highest order of the sciences (Tolan, 2023).

Swineshead's magnum opus was a series of treatises known as the Liber calculationum ("Book of Calculations"), written c. 1350, which earned him the nickname of The Calculator. Leibniz had a copy of one of Swineshead's treatises made from an edition in the Bibliothèque du Roi in Paris and credited Swineshead as "the man 'who introduced mathematics into scholastic philosophy (Rescher, 2013)..

Oresme was first a pupil, then a teacher, and finally introduced to the Collège de Navarre in Paris. He died as Bishop of Lisieux. He developed learning about the intensity of forms and understood variable quantity as intensity or degree. Functional dependency was described verbally by a rule or graphically. For the dependency of quantities, he used the term proportio (ratio, relationship) in his tract *Tractatus proportionum* and *Algorithm proportionum, writing*: "*... any relationship would be shown between one intensity and another, the same relationship is found between the first and second lines...*" (Juškevič, 1970, p. 141).

Among scholars, the idea of the laws of nature as laws of the functional type began to emerge, and various more general theories of change of quantity as a function of time arose.

2 New Age

At the turn of the 16th and 17th centuries (Juškevič, 1972), Scottish mathematician, physicist, astronomer and astrologer John Napier (1550-1617) discovered logarithms that became a suitable tool for large numerical calculations in astronomy and maritime navigation. Looking at his approach to logarithms, this proves once again that the concept of physical movement was still the basis for thinking about quantitative changes to the continuous variable.

In the 17th century (Cantor, 1908), mathematics experienced a transition from constant quantities and numbers to variable quantities, and was inspired by technical problems. It can now be said that the concept of (predominantly natural) regularity was beginning to grow into the concept of functional dependency. The profession of "pure" mathematics did not exist at the time; the mathematician was also a physicist, stargazer, astrologer, mechanic, doctor or philosopher. Newly formed analytical geometry (Pierre de Fermat (1601 – 1665) and René Descartes (1596 – 1650)) allowed a description of the dependency between two variable equations with unknown x and y use of coordinates (from today's point of view it was an implicit input of a function where it was often possible to express one variable by the other). Descartes began to use the terms independently variable and dependent variable.

Although the concept of function was not defined, Isaac Newton (1643 – 1727) and Gottfried Wilhelm Leibniz (1646 – 1716) laid the foundations for differential and integral calculus. Newton's reasoning was based on a description of the movement of the material point, and Leibniz was inspired by the search for the tangent to the curve. Various infinitesimal considerations and calculations appeared in mathematics as early as antiquity (Rescher, 2013).

Before Newton and Leibniz they were followed by Johannes Kepler (1571 – 1630), Fermat, Blaise Pascal (1623 – 1662), Gilles Roberval (1602 – 1675) and others, but Newton and Leibniz carried out their methods into generally applicable algorithms and were also the first to recognise the connection between the tangent and the quadrature (Juškevič, 1972).

Newton in his paper *Method of Fluxions and Infinite Series* (1671, published only in r. 1736 after his death) uses the concept of fluentes quantitates, which are indeterminate time-dependent quantities that continue to grow or decrease at the same time as the formation of curves by the influence of local motion, and fluxio, indicating the rate of growth or descent. Later, Newton generalizes the naming of the two quantities and abstracts to some extent from time dependence: fluenta is called quantitas correlata, fluxi quantitas relata. It admits to

expressing fluenty analytically or in the form of the sum of an infinite series, hence "its" functions are actually polynomials (Cantor, 1908).

Leibniz bases his conception of differential calculus on the idea of a characteristic triangle with the sides d*x*, d*y*, and d*s* (thus generalizing Pascal's earlier idea), Descartes's analytical expression of geometric curves, and the theory of infinite series (Cantor, 1908).

In. 1673 Leibniz uses the term function in the manuscript *Methodus tangentium inversa, seu de functionibu*s for the first time, but the word function has a different meaning than today. The name is based on geometry, the concept is closely bound to the curve and its tangential at a given point of the curve, functions calling the various "parts of lines and lines" formed in the construction of this tangent. The word function is used in two ways: as a certain property or role, as a naming of a result or an object with this property. It is likely that Leibniz's understanding of this new mathematical term corresponds to the Latin meaning of the word: fungor means to perform, to perform, to be active, and functio signifies activity, to perform (Cantor, 1908, Edwards, 1979).

In the following decades, the word function appeared several times in the correspondence of mathematicians, generally in the original Leibniz narrow meaning, but in r. 1694 Johann Bernoulli (1667 – 1748) indicates by letter *n* any quantity, however, arising from indeterminate and constants (*posito n esse quantitatem quomodocunque formatam ex indeterminatis ex constantibus*), but does not use the word function (Juškevič, 1972, p. 146). A major encyclopedic work, *Mathematisches Lexicon*, published in r. 1716 in Leipzig prof. Wolff, the word function makes no mention, there is only a chapter on constant and variable (Cantor, 1908, Edwards, 1979).

He first formulated the definition of the function in r. 1718 Johann Bernoulli in mémoires de l'Academie des Sciences de Paris: "*He apelle ici fonction d'une gradeur variable, une quantité composée de quolqur maniére que ce soit de cette grandeur variable et de constants*“ (i.e.: A variable function is called a quantity compiled in any way from this variable and constants – Cantor, 1908, p. 457, Juškevič, 1972, p. 147).

This definition marked the first attempt to introduce a new concept in regulation. What functions did mathematicians have in mind in the 18th century? The function was seen mostly geometrically as a curve (usually one that could be sketched with the free movement of a hand without raising a hand from the paper, which was then a criterion of connection) or geometric-kinematically as a path of a moving point.

Let's state how the concept of function was perceived by some other eminent mathematicians, as their view gradually expanded and the definition of the function continued to become generalized so that the content of the definition replicated its then known scope.

Leonhard Euler (1707 – 1783) defined the function of variable quantity in r. 1748 first, the words "*analytical expression compiled in any way from this variable and numbers or constant quantities*" (Church, 1956, p. 22, Juškevič, 1972, p. 250), conceding complex numbers for independent variables. He considered the quantity compiled (compositum) from the second quantities to be one that could be obtained from these quantities by four elementary operations, then by the square root, or by any conceivable operation. Extensive discussions about the content of the concept of function held between mathematicians in the 18th century and the small generality of the above definition led in r. 1755 Euler on the formulation of a more general definition of the function in which he emphasized the dependency of quantities and not the mode of expression: "*If some quantities depend on others so that they themselves are subject to change when they change them, then we call the former the functions of the latter. This name is extremely broad in nature; it covers all the ways in which one quantity can be expressed by others*." (Juškevič, 1972, p. 254)

It is worth remembering that for Euler, for example, the $x \dashrightarrow \frac{1}{x}$ function was continuous everywhere because it is given by a single regulation, whereas the $x \dashrightarrow |x|$ function, whose analytical expression he understood to be two regulations: $x \dashrightarrow x$ (if $x \geq 0$) and $x \dashrightarrow -x$ (if $x < 0$), was considered by Euler to be disjointed. Let's state that the current view of the connection is different. The association is a local feature of the function and is not decisive at all. Whether it is possible to describe the function in a single analytical expression throughout the definition field, or whether the expression exists at all. The function was understood a priori in the 16th century as a changing quantity and e.g. the constant Euler did not rank among the functions (Juškevič, 1972).

We will note the further development of the concept of the function very briefly, because in the development of the concept of the function is, in our opinion, primarily the period of birth and the first discussions about the concept of the function terminated by the now classical concept of the function in Dirichlet and Lobachevsky (see Juškevič, 1972, Vopěnka, 2010).

In the 19th century, for example, G contributed significantly to further discussions on the content of the concept of function. F. Bernhard Riemann (1826 – 1866), Peter Gustav L. Dirichlet (1805 – 1859), Augustin Louis Cauchy (1789 – 1857), Bernard Bolzano (1781 – 1848), who contributed, among other things, by enriching mathematics with some

"pathological" functions; Sylvestre F. Lacroix (1765 – 1843), Joseph Fourier (1768 – 1830) and especially Nikolai Ivanovich Lobachevsky (1793 – 1856), who in 1856. 1834 for the first time emphasised the necessity of the unequivocality of the variable. It is Lobachevsky and Dirichlet's conception of the function as the unambiguous attribution of the dependent variable of the independent variable that is considered classic nowadays (Vopěnka, 2010, 2011).

By Cantor (1908) Lobachevsky (1834) writes that

The general concept of a function requires that a function of *x* be defined as a number given for each *x* and varying gradually with *x*. The value of the function can be given either by an analytic expression, or by a condition that provides a means of examining all numbers and choosing one of them; or finally the dependence may exist but remain unknown.

while (see Cantor, 1908) Dirichlet (1837) writes

If now a unique finite *y* corresponding to each *x*, and moreover in such a way that when *x* ranges continuously over the interval from *a* to *b*, $y = f(x)$- also varies continuously, then *y* is called a *continuous* function of *x* for this interval. It is not at all necessary here that *y* be given in terms of *x* by one and the same law throughout the entire interval, and it is not necessary that it be regarded as a dependence expressed using mathematical operations.

Emerging set theory at the turn of the 19th and 20th centuries allowed for the view of functions as the depiction of one set into another. Richard Dedekind (1831 – 1916) in R. While 1887 defines (see Coufal et all, 2017, Dedekind, 1969, Edwards, 1979, Jarník, 1981) the concept of imaging as a system, it does not put it at all in relation to the traditional concept of the function. This connection is emphasised more in Cantor (1908) by Georg Cantor (1845 – 1918) or Constantin Carathéodory (1873 – 1950) (see Vopěnka, 2010). In the 19th century, the demands of the rigorous development of analysis by Karl Weierstrass and others, the reformulation of geometry in terms of analysis, and the invention of set theory by Georg Cantor, eventually led to the much more general modern concept of a function as a single-valued mapping from one set to another.

David Hilbert (1862 – 1943) set himself the goal of "formalizing" classical mathematics "as a formal axiomatic theory, and this theory shall be proved to be consistent, i.e., free from contradiction". In Hilbert 1927 The Foundations of Mathematics he frames the notion of function in terms of the existence of an "object" (see Bourbaki, 1994)

Many mathematicians resented the fact that traditional definitions of the function contained undefined terms such as law, rule, relationship, assignment, correspondence, operations, but also a new set and display, demanding to reduce the number of undefined terms. Giuseppe Peano (1858 – 1932), for example, did so with a single undefined concept of a set

and understood the function as a special binary relation, i.e. as a set of arranged pairs. A distinctly structuralist approach to the job was taken by a group of French mathematicians under the pseudonym Nicolas Bourbaki (1994).

The difficulty of defining such a general concept, which functions were particularly pointed out by some logicians, e.g. in Church (1956) Alonzo Church (1903 -1995), which proposed introducing the function as a primitive concept.

Conclusion

What was the function in its historical development, how did it manifest itself in mathematics, and what terms appeared in its development? A look at the history of the concept's evolution shows that the function was a quantity, a formula, an analytical expression, a sequence of values, but also a dependency, intensity, relationship, (binary) relation, or set of (arranged pairs), was perceived as a curve or path of a moving body or point, viewed as a primitive, undefined concept. Instead of a direct definition of the function, the so-called functional situation under which the function is given has sometimes been described.

If you were talking about a function, you could mean a table of discrete values, a verbal description, a graph, a formula (one or several), or an equation of two variables. In relation to the function, geometric and kinematic terminology often intersected. As a function, time-dependent events were often studied (especially initially), implying a connection, the non-negative ness of the argument, very often smoothness.

The idea of continuous passage through time (according to Newton, fluentity) was initially very strong, in addition to which immutability was perceived as something pathological and constant as a function was not considered. The phenomenon of dependency and variability was dominant in relation to the function, the requirement of unequivocality of functional value did not begin to appear in the definitions until later.

With a brief historical overview and highlighting some of the influences associated with the development of the concept of function, we wanted to draw attention to the great variety of views on this concept and point to the possible sources of thought applicable in the teaching of mathematics.

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